Transition chains in the planar restricted elliptic three body problem

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Plan of the presentation:

- The dynamics in the PRC3BP for energies close to the energy of L_2.

- The PRE3BP as a perturbation of the PRC3BP

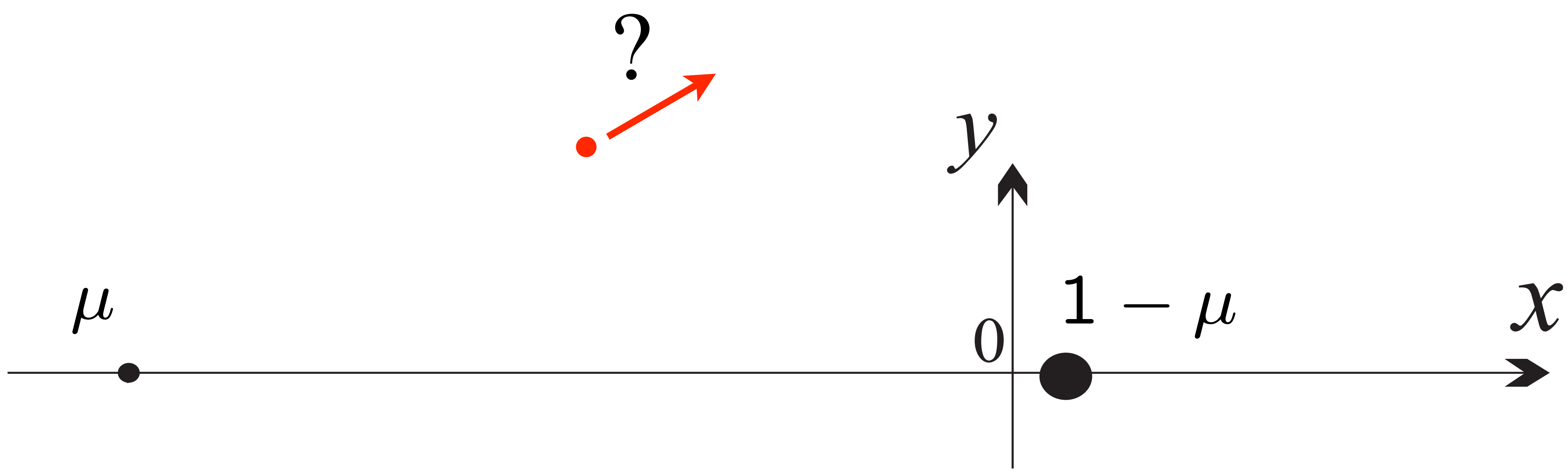
- Lapunov-Moser Theorem and the twist property on the set of Lapunov orbits.

- KAM Theorem and the persistence of Lapunov orbits

- A Melnikov type method and transversal intersections of invariant manifolds of KAM tori.

- Concluding remarks

The Planar Restricted Circular Three Body Problem (PRC3BP)



$$H(x,y,p_x,p_y) = \frac{(p_x+y)^2+(p_y-x)^2}{2} - \Omega(x,y),$$

$$\Omega(x,y) = \frac{x^2+y^2}{2} + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}.$$

$$H(x, y, p_x, p_y) = \frac{(p_x + y)^2 + (p_y - x)^2}{2} - \Omega(x, y)$$

Jacobi integral:

$$F(x, y, p_x, p_y) = -2H(x, y, p_x, p_y)$$

The constant energy manifold:

$$M(C) = \{(x, y, p_x, p_y) \in \mathbb{R}^4 | F(x, y, p_x, p_y) = C\}$$

Hill's region:

$$R(C) = \{(x, y) \in \mathbb{R}^2 | \Omega(x, y) \geq C/2\}$$

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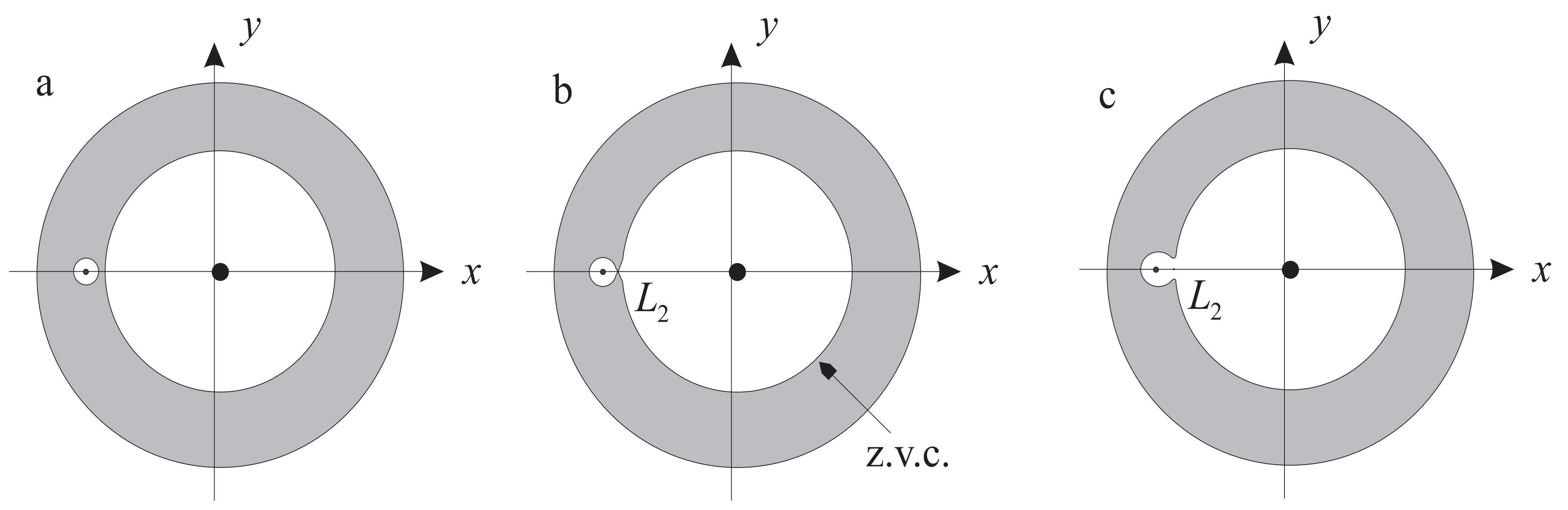
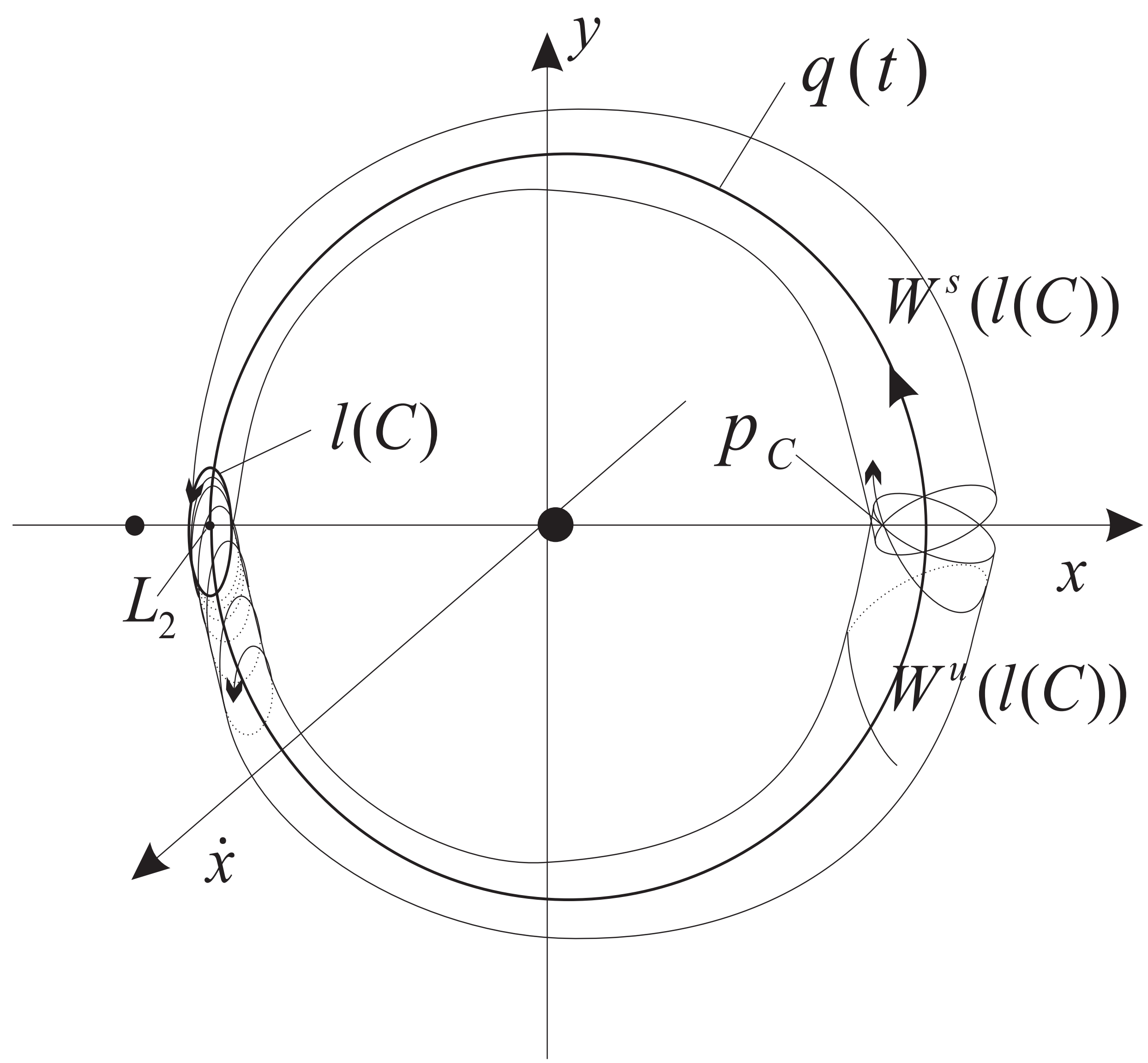


Various shapes of the Hill's region

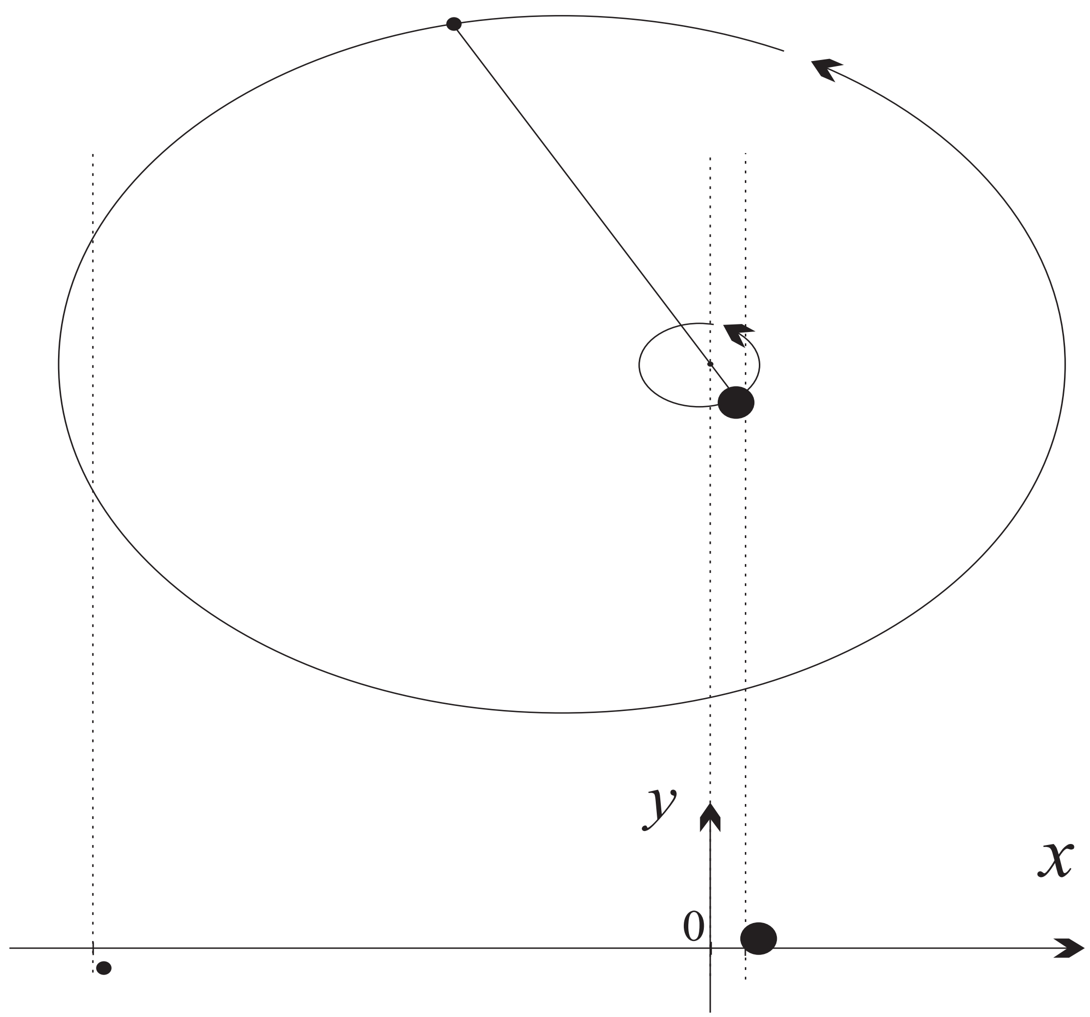
a. $C > C_2$ b. $C = C_2$ c. $C < C_2$

The dynamics in the PRC3BP

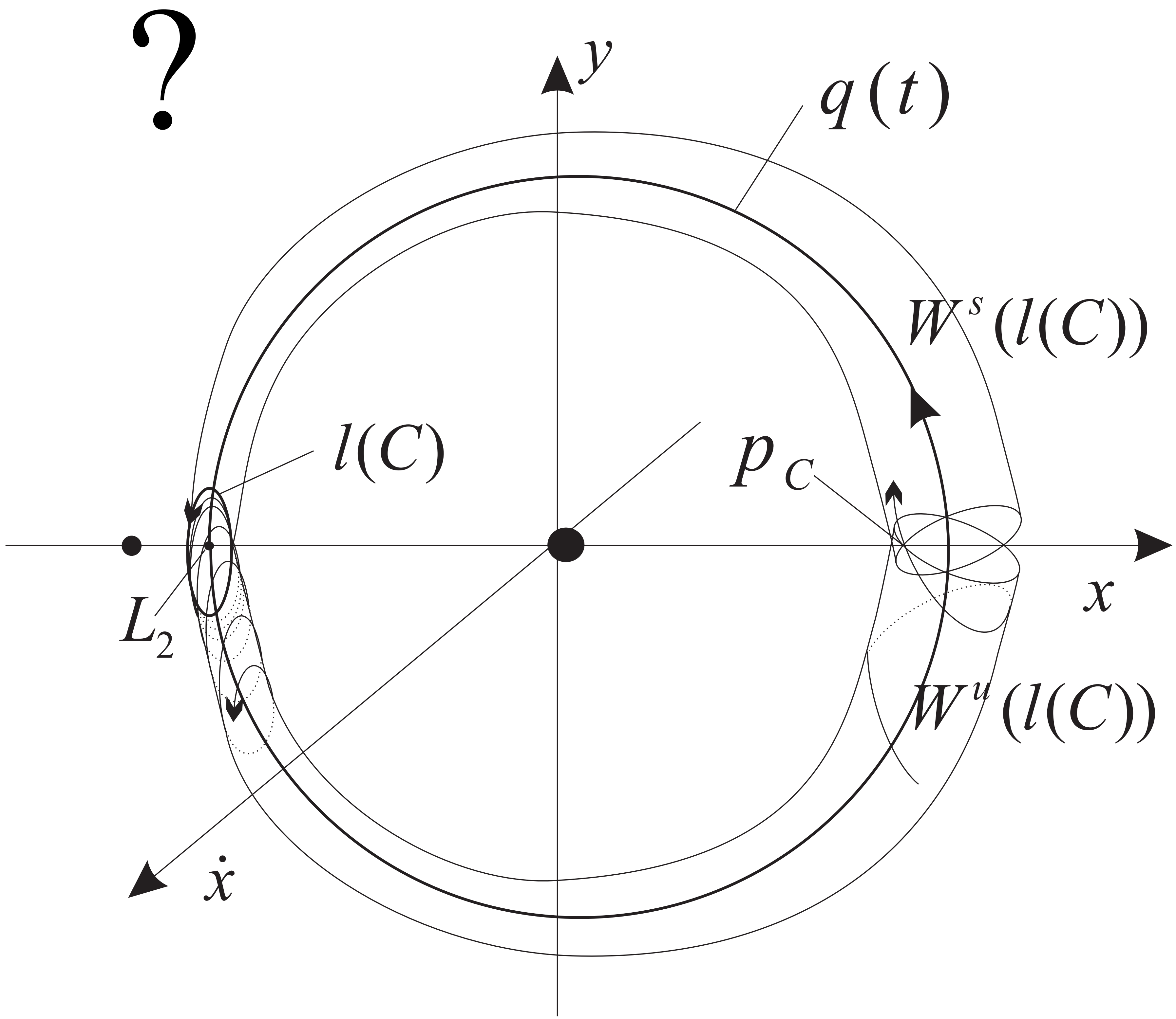


[LMS] J. Llibre, R. Martinez, C. Simo, *Tansversality of the Invariant Manifolds Associated to the Lyapunov Family of Periodic Orbits Near* L_2 *in the Restricted Three Body Problem*, Jour. of Diff. Eq. 58, 104-156 (1985).

The equations for the PRE3BP



$$H^e(x, y, \dot{x}, \dot{y}, t) = H(x, y, \dot{x}, \dot{y}) + eG(x, y, t) + O(e^2)$$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W^s(l(C))
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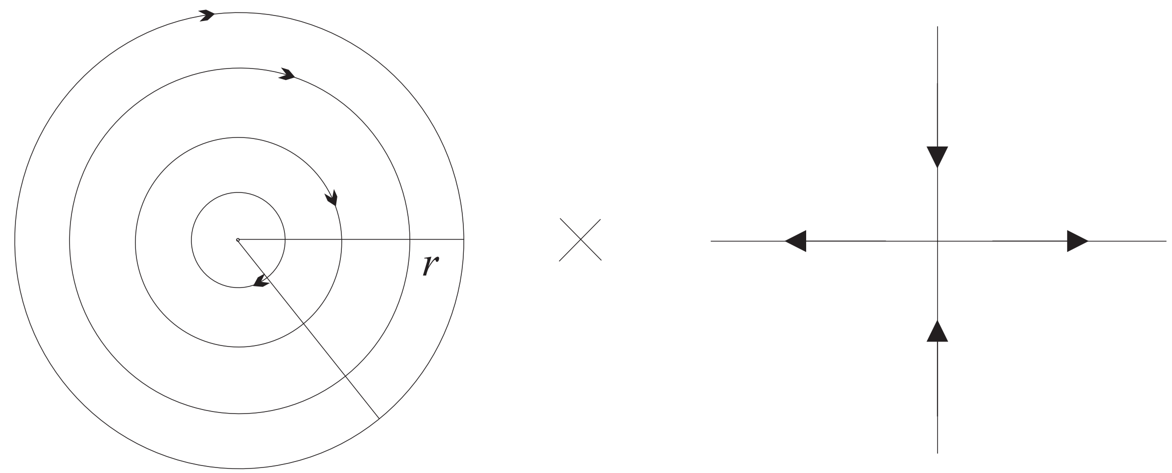


$$\begin{aligned} l(t) &= re^{ta_2(ir^2)+i\phi} \\ a_2(ir^2) &= \alpha_2 + ir^2\mathbf{a}_2 + \ldots \end{aligned}$$

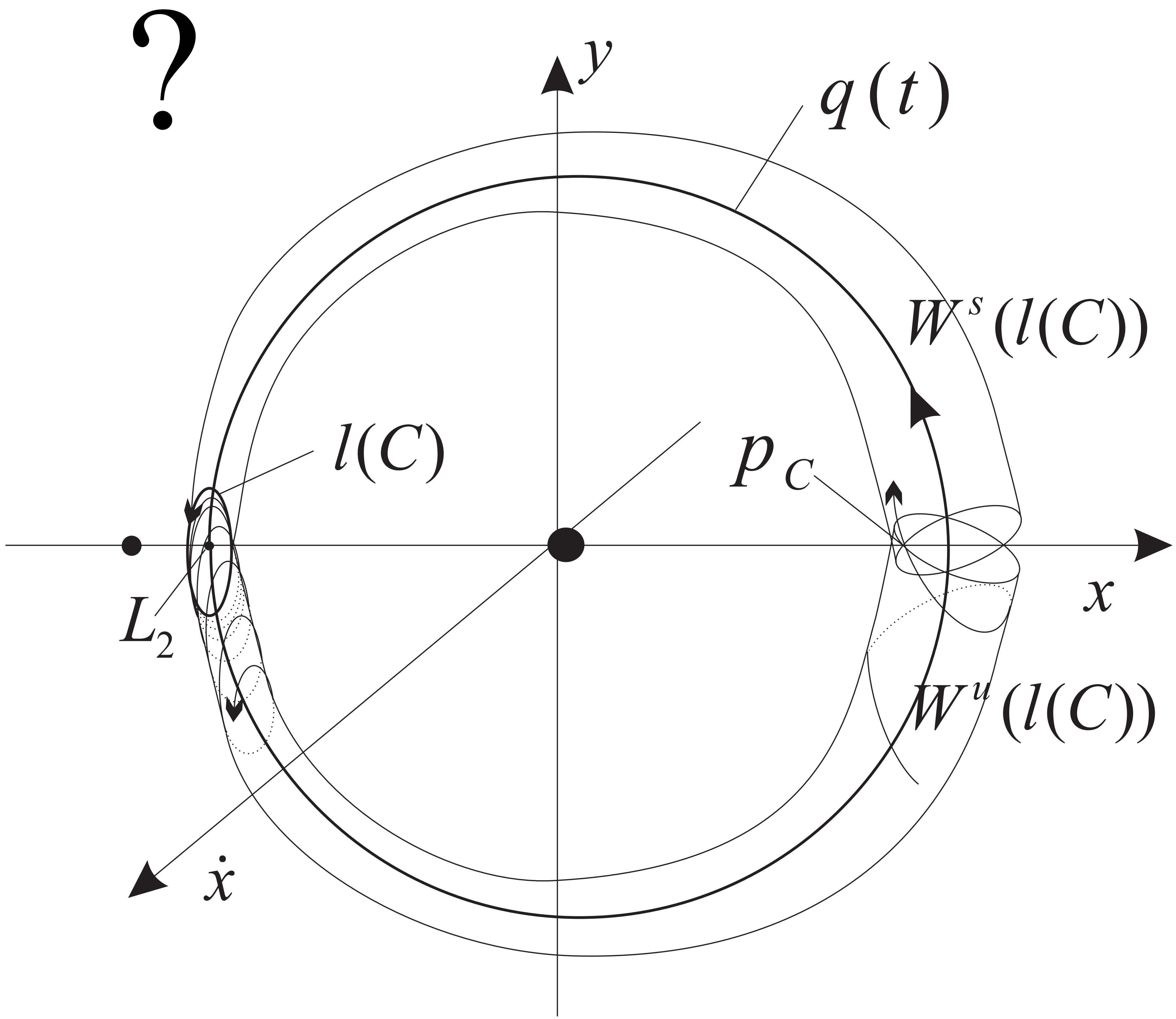
Lemma 1 *If the coefficient* $\mathbf{a}_2$ *is nonzero then on the set of Lapunov orbits with a sufficiently small radius* r *the time* 2π *shift along the trajectory is a twist map.*

[M] J. Moser, *On the Generalization of a theorem of A. Liapounoff*, Communications on Pure and Applied Mathematics, Vol. xi (1958), 257-278.

Lemma 2 *For sufficiently small masses* μ *the twist coefficient* $\mathbf{a}_2(\mu)$ *of the PRC3BP is approximated by the twist coefficient* $\mathbf{a}_2^{Hill}$ *of the Hill's problem*

$$\lim_{\mu\to 0} \mu^{2/3}\mathbf{a}_2^{\mu} = \mathbf{a}_2^{Hill}.$$

$$\mathbf{a}_2^{\mathrm{Hill}} = \frac{\frac{2187}{16}(1-2\sqrt{7})3^{2/3}(5767\sqrt{7}-15274)}{(1+2\sqrt{7})^2(\sqrt{7}-3)^2(4\sqrt{7}-7)(\sqrt{7}-14)^2} \approx 8.483$$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W^s(l(C))
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W^u(l(C))
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Theorem 3 *For a sufficiently small mass* $\mu > 0$ *there exists a Cantor set of energies* $\mathfrak{C}$ *such that for any* $c \in \mathfrak{C}$ *the Lapunov orbit* $l(c)$ *is perturbed into an invariant torus* $l^e_{t_0}(c)$ *of the time* 2π *shift along the trajectory map* $P^e_{t_0}$ *of the PRE3BP*

$$P^e_{t_0} : \Sigma_{t_0} \to \Sigma_{t_0+2\pi}$$

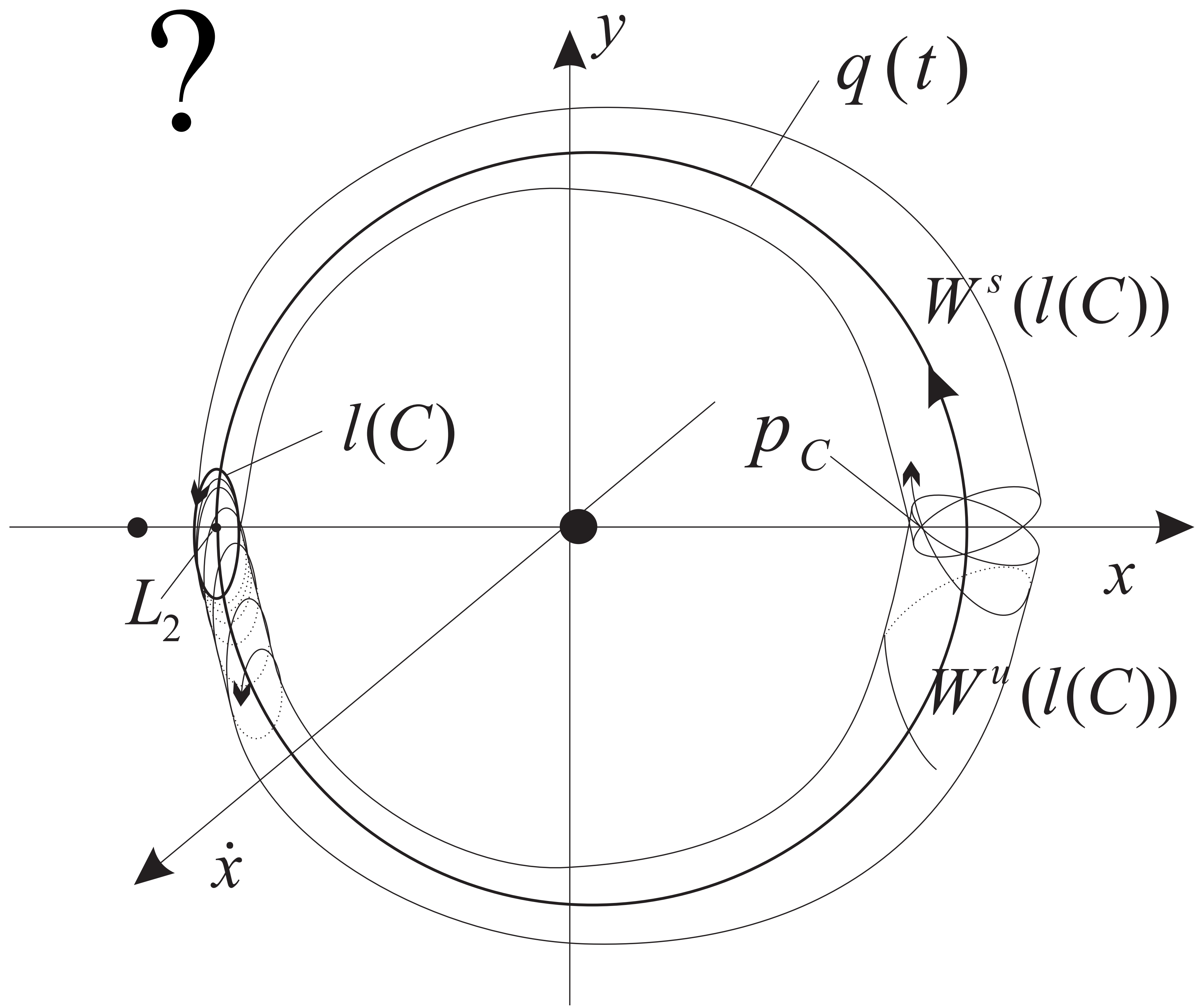
where $\Sigma_{t_0} = \{(x, y, p_x, p_y, t) | t = t_0\}$.

Outline of the proof:

- The set of Lapunov orbits is normally hyperbolic and persists under the perturbation.

- Before the perturbation the map $P_{t_0}^{e=0}$ is a twist map which allows us to apply the KAM Theorem to obtain our result.

[DLS] A. Delshams, R.de la Llave, T. Seara, *A geometric approach to the existence of orbits with unbounded energy in generic periodic perturbations by a potential of generic geodesic flows of* $\mathbb{T}^2$. Comm. Math. Phys. 209 (2000), no. 2, 353–392.


?
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x
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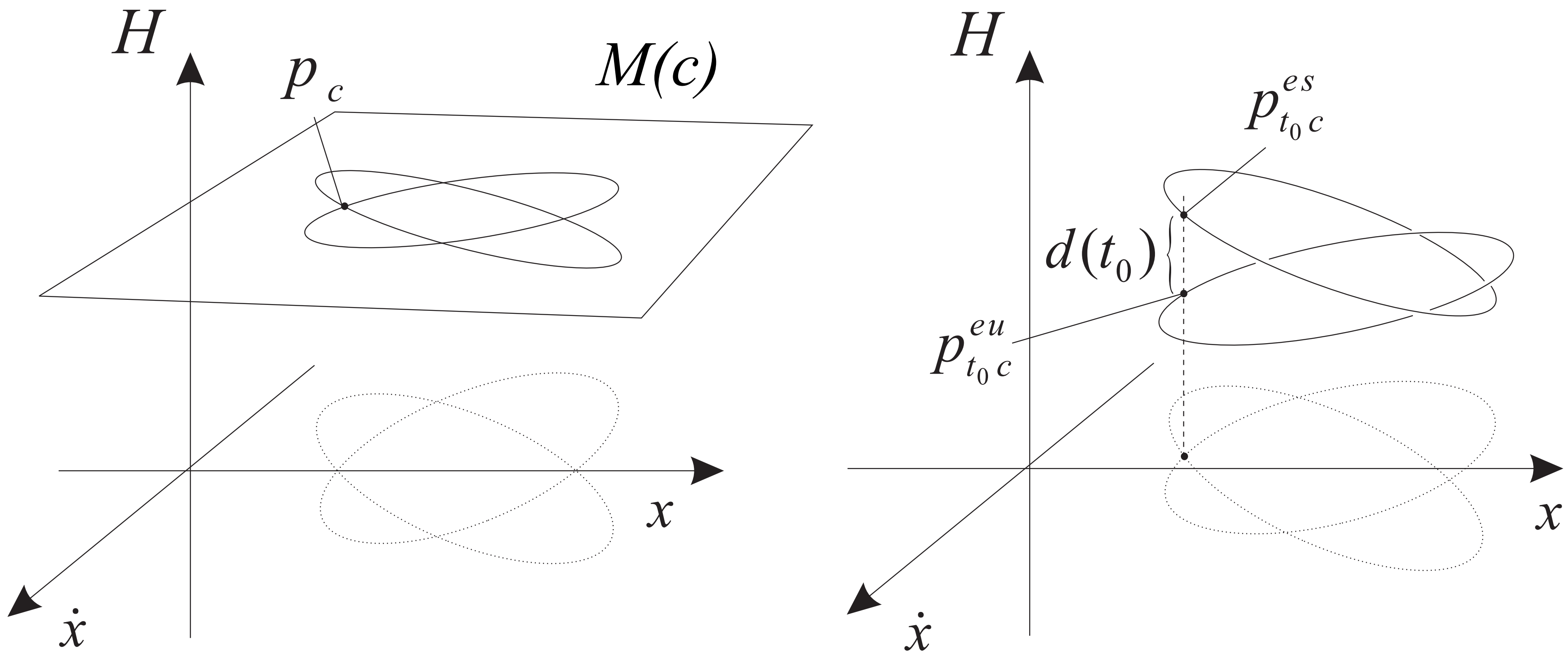


The intersection of the invariant manifolds of the PRC3BP compared with the situation in the PRE3BP.

$$P^e_{t_0} : \Sigma_{t_0} \to \Sigma_{t_0+2\pi}$$

$$H^e(\mathbf{x}, t) = H(\mathbf{x}) + eG(\mathbf{x}, t) + O(e^2)$$

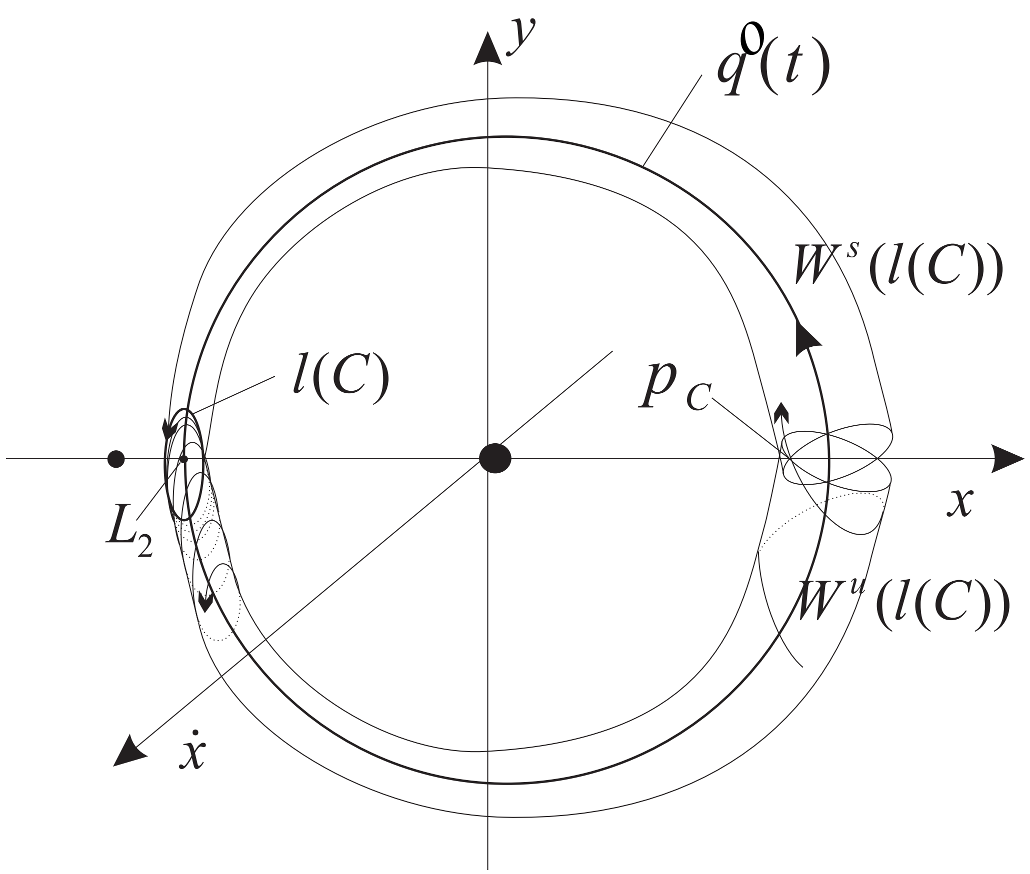
$q^0(t)$ is the homoclinic orbit to L_2

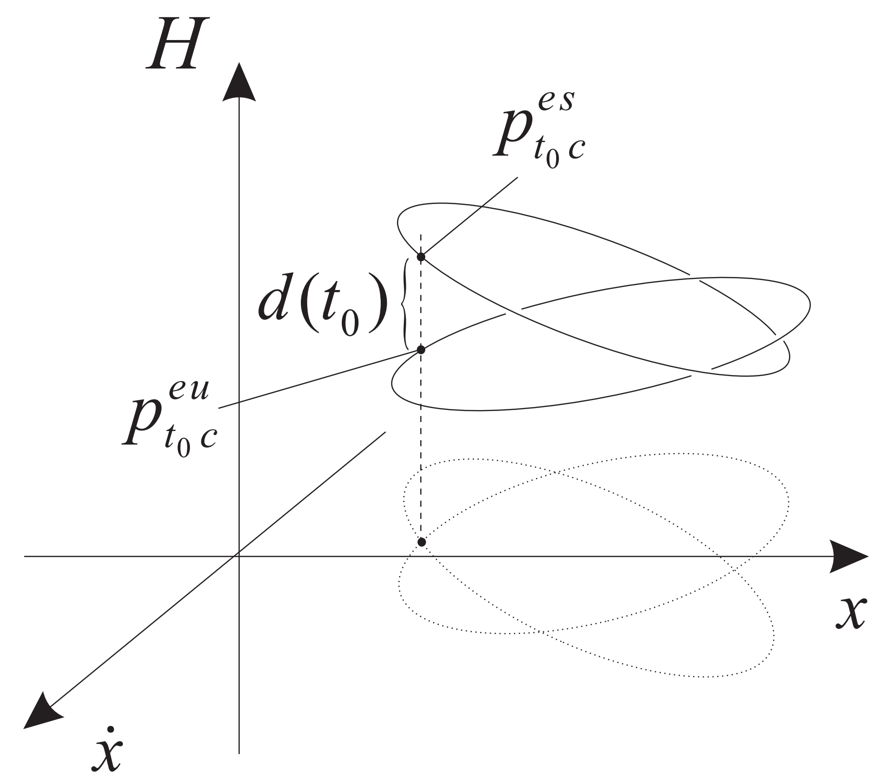
Theorem 4 *If*

$$M(t_0) = \int_{-\infty}^{+\infty} \{H, G\}(q^0(t - t_0), t)dt$$

has simple zeros then for $c < C_2$ sufficiently close to C_2 the manifolds $W^s(l^e_{t_0}(c), P^e_{t_0})$ and $W^u(l^e_{t_0}(c), P^e_{t_0})$ intersect transversally.

Some remarks about the method





$$\int_{t_0}^{+\infty} \{H, G\}(q^0(t - t_0), t)dt$$

$$d(t_0) = eM(t_0) + O(e\Delta c) + o(e)$$

$$\Delta c = \sqrt{C_2 - c}$$

Computation of the Melnikov function

$$M(t_0) = \int_{-\infty}^{+\infty} \{H, G\}(q^0(t - t_0), t)dt$$

$$\begin{aligned} R(x, y, p_x, p_y, t) &:= (x, -y, -p_x, p_y, -t) \\ R(q^0(t), t) &= \left(q^0(-t), -t\right). \end{aligned}$$

$$\{H, G\}(R(x, y, p_x, p_y, t)) = -\{H, G\}(x, y, p_x, p_y, t).$$

Lemma 5 *For $t_0 = 0$ the Melnikov integral is equal to zero $M(0) = 0$.*

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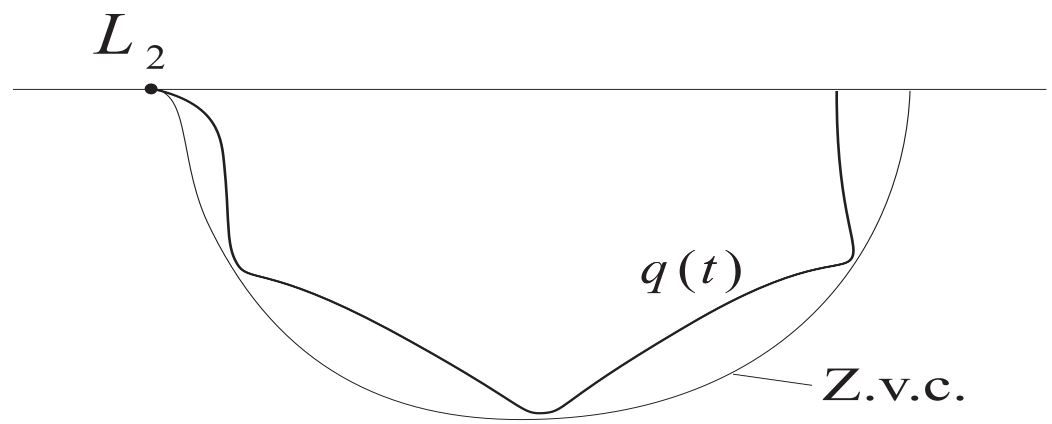
Proof

$$\begin{aligned}\int_{-\infty}^{0}\{H,G\}(q^0(t),t)dt &= \int_0^{+\infty}\{H,G\}(q^0(-t),-t)dt \\ &= \int_0^{+\infty}\{H,G\}(R(q^0(t),t))dt \\ &= -\int_0^{+\infty}\{H,G\}(q^0(t),t)dt\end{aligned}$$

$$M(t_0) = \int_{-\infty}^{\infty}\{H,G\}(q^0(t-t_0),t)dt.$$

$\square$

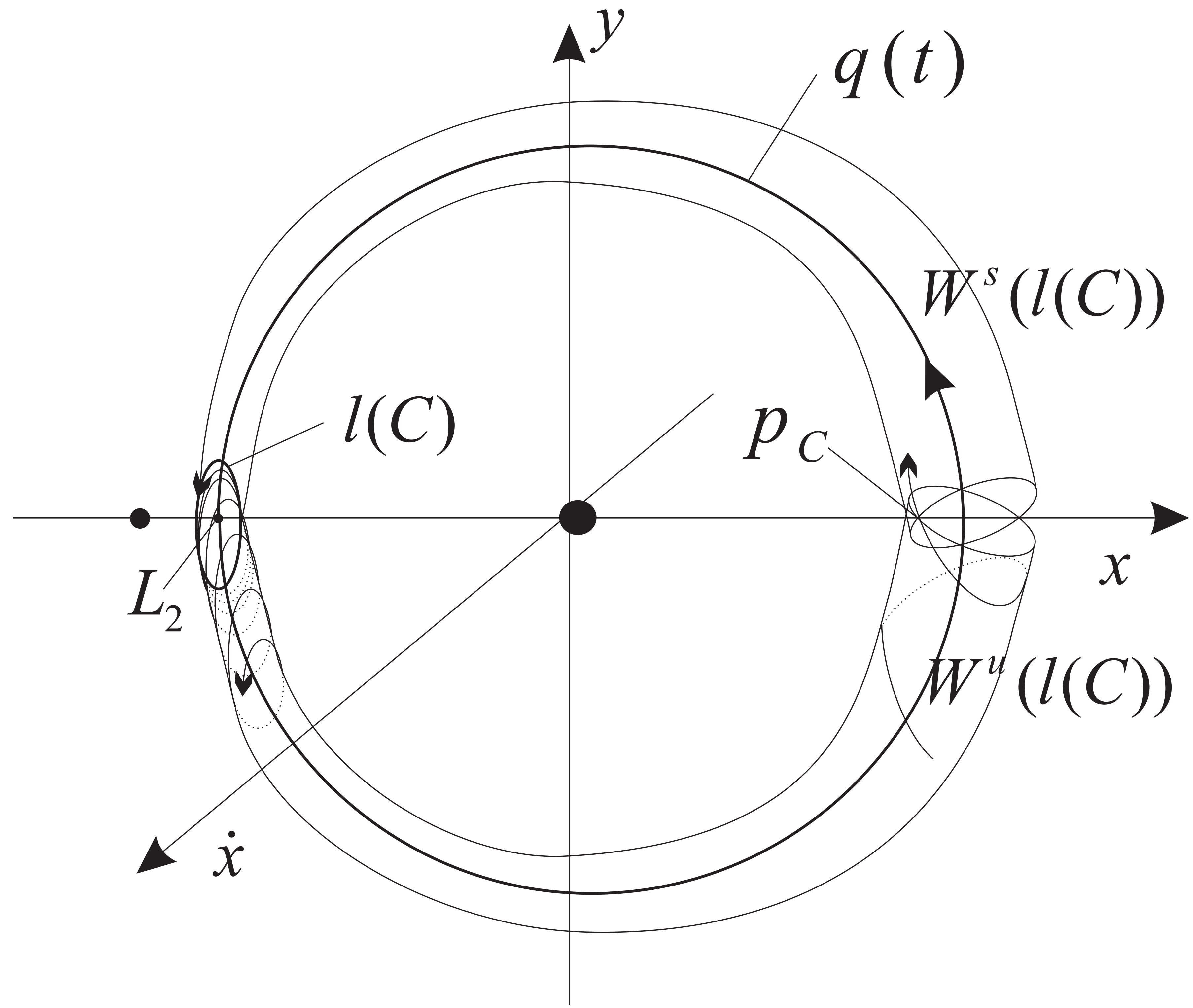
Computation of $M_{t_0}(0)$



Lemma 6 *The derivative of the Melnikov function can be approximated by an integral over the unstable orbit* $q^H(t)$ *of the Hill's problem*

$$\lim_{\mu\to 0}|M_{t_0}(0)| = \left|\int_{-\infty}^{\infty} 2\{H,G\}(q^H(t))dt\right|.$$

$$\left|\int_{-\infty}^{\infty}\{H,G\}(q^H(t))dt\right| \approx 2.06$$



y
$q(t)$
$W^s(l(C))$
$l(C)$
p_C
x
L_2
$W^u(l(C))$
$\dot{x}$

Theorem 7 *For sufficiently small $e > 0$ and for sufficiently small masses $\mu > 0$ there exists a Cantor set of Lapunov orbits which persists from the PRC3BP to the PRE3BP. What is more the corresponding stable and unstable manifolds of the perturbed orbits intersect transversally and form transition chains. Each transition involves a change of the energy.*

Concluding remarks

- The result holds only for energies sufficiently close to C_2 and for sufficiently small e and μ.

- No bounds on these parameters are given.

- The discussed model has no direct examples in celestial mechanics.

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Future research?

- It would be desirable to perform a similar argument for the Jupiter - Sun system.

- Major obstacle (?): Application of the KAM theorem for a given eccentricity and for given radius of the Lapunov orbit .