Rigorous numerics for infinite dimensional maps

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joint work with
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Line of reasoning

infinite dimensional map

$\downarrow$Galerkin + truncation estimate

finite dimensional multivalued map

$\downarrow$spatial discretization (GAIO)

combinatorial multivalued map (directed graph)

$\downarrow$graph algorithms

combinatorial index pair

$\downarrow$computational homology (CHomP)

Conley index for finite dimensional continuous selector

$\downarrow$lifting

Conley index for original map

The map

The Kot-Schaffer growth-dispersal model for plants:

$$\Phi : L^2 \to L^2, \quad \Phi(a)(y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} b(x,y) \ \mu \, a(x) \left(1 - \frac{a(x)}{c(x)}\right) dx,$$

$a, b, c \in L^2([-\pi, \pi]), \mu > 0, b(x,y) = b(x-y).$

Equivalent countable system

Using a basis of Fourier-modes $\varphi_k = \exp(ik\cdot)$ for L^2 one gets the countable system of maps:

$$f_k(a) = \mu b_k \left[a_k - \sum_{j+l+n=k} c_j a_l a_n \right], \quad k \in \mathbb{Z},$$

a_k, b_k, c_k Fourier coefficients of a, b, c^{-1}.

Line of reasoning

- Let $P_m : L^2 \to X_m = \text{span}\{\varphi_0, \dots, \varphi_{m-1}\}$ be the projection onto the first m modes and consider the finite dimensional map

$$f^{(m)} : X_m \to X_m, \quad f^{(m)} = P_m \circ f;$$

- What is the relation between the dynamics of f and of $f^{(m)}$?

- Write

$$f(a) = f(P_m a) + (f(a) - f(P_m a))$$

and suppose that we can bound $f(a) - f(P_m a)$ on a compact subset

$$Z = W \times V, \quad W \subset X_m,$$

of L^2:

$$|f(a) - f(P_m a)| < \varepsilon^{(m)} \quad \text{for all } a \in Z.$$

- Now consider a *multivalued* map $F^{(m)} : W \rightrightarrows X_m$ with the property that for all $a \in Z$

$$P_m f(a) \in F^{(m)}(P_m a).$$

- Compute objects of interest for $F^{(m)}$ via a rigorous set-oriented approach in combination with the Conley-index theory:

  - cover the *maximal invariant set* of $F^{(m)}$ in W;

  - compute approximate locations of *objects of interest* (periodic points, connecting orbits, chain recurrent sets);

  - construct an *isolating neighborhood* and an *index pair* of the desired invariant set;

  - compute its *Conley index*;

- *Lift the information* on $F^{(m)}$, resp. $f^{(m)}$, to the full system Φ.

Finite dimensional multivalued map

$$F_k^{(m)}(a_0,\ldots,a_{m-1}) = \mu b_k \left[a_k - \sum_{\substack{j+l+n=k \\ 0\le j,l,n\le m-1}} c_j a_l a_n \right] + \varepsilon_k^{(m)}\,[-1,1]\,,$$

$k = 0, 1, \ldots, m-1.$

The error $\varepsilon_k^{(m)}$ has been computed in such a way that

$$\left| f_k(a) - f_k^{(m)}(a_0,\ldots,a_{m-1}) \right| \in \varepsilon_k^{(m)}[-1,1]$$

for all a in some compact set $Z = W \times V \subset L^2$.

Objects in Phase Space

- A *full trajectory* of F is given by $\sigma : \mathbb{Z} \to X, \quad \sigma(n+1) \in F(\sigma(n))$;
- A set $S \subset W$ is *invariant*, if for every $x \in S$ there exists a full trajectory $\sigma : \mathbb{Z} \to S$ with $\sigma(0) = x$.
- The *maximal invariant set* of a subset S is given by

$$\text{Inv}(S, F) = \{x \in S \mid \exists \sigma : \mathbb{Z} \to S, \sigma(0) = x\}.$$

- An *isolating neighborhood* is a compact set $I \subset W$ such that

$$\text{Inv}(I, F) \subset \text{int}(I).$$

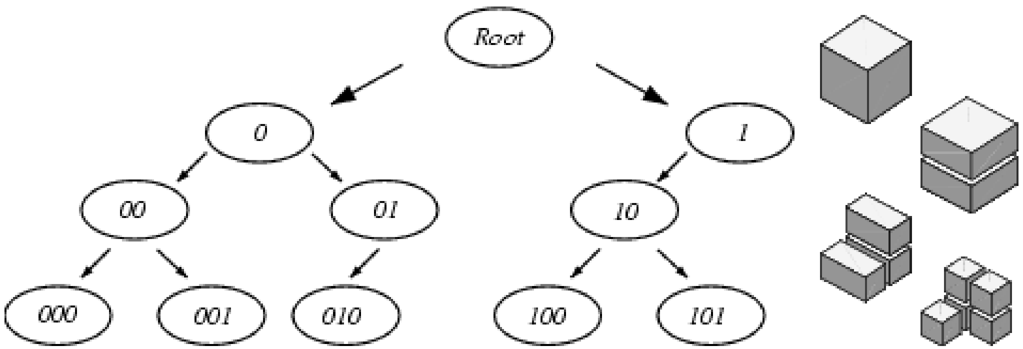
  An invariant set is *isolated*, if it is the maximal invariant set of some isolating neighborhood.

Spatial Discretization

- Goal: *global* analysis of F (i.e. computation of invariant sets);
- partition (part of) W into a finite grid $\mathcal{B} = \{B_1, \ldots, B_b\}$ of compact connected sets, $W = \bigcup_{i=1}^{b} B_i$
- and define a multivalued map $\mathcal{F} : \mathcal{B} \to \mathcal{B}$ by

$$\mathcal{F}(B) = \left\{ B' \in \mathcal{B} \mid F(B) \cap B' \neq \emptyset \right\};$$

- Implementation: $\mathcal{B}$ in a *binary tree*



$\mathcal{F}$ as a *(sparse) matrix* $\equiv$ *directed graph*.

Objects for $\mathcal{F}$

- The notions of *trajectory*, *invariant set* and *maximal invariant set* directly carry over to $\mathcal{F}$.
- For $\mathcal{S} \subset \mathcal{B}$ let $|\mathcal{S}|$ denote the union of the elements in $\mathcal{S}$ and let

$$o(\mathcal{S}) = \{B \in \mathcal{B} \mid B \cap |\mathcal{S}| \neq \emptyset\}$$

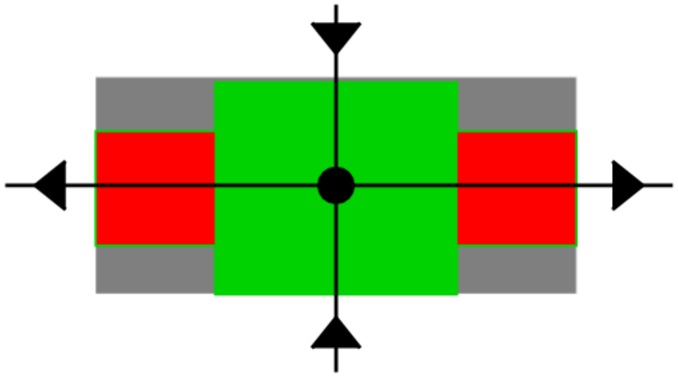
  be the *smallest representable neighborhood* of $\mathcal{S}$.
- A *(combinatorial) isolating neighborhood* for $\mathcal{F}$ is a set $\mathcal{I} \subset \mathcal{B}$ such that

$$o(\mathrm{Inv}(\mathcal{I}, \mathcal{F})) \subset \mathcal{I}.$$

Proposition 1 *If $\mathcal{I}$ is an isolating neighborhood for $\mathcal{F}$, then $|\mathcal{I}|$ is an isolating neighborhood for F.*

Index pairs

Let $\mathcal{I}$ be an isolating neighborhood for $\mathcal{F}$. A pair $\mathcal{N} = (\mathcal{N}_1, \mathcal{N}_0)$, $\mathcal{N}_0 \subset \mathcal{N}_1 \subset \mathcal{I}$ is an *index pair* if



Theorem 1 (Szymczak, 97) *Let $\mathcal{S}$ be an isolated invariant set for $\mathcal{F}$ and let*

$$\mathcal{N}_1 = \mathcal{S} \cup \mathcal{F}(\mathcal{S}), \quad \mathcal{N}_0 = \mathcal{N}_1 \backslash \mathcal{S}.$$

Then $\mathcal{N} = (\mathcal{N}_1, \mathcal{N}_0)$ is an index pair.

Computing Isolating Neighborhoods

- Consider the transition matrix

$$P = (p_{ij}), \quad p_{ij} = \begin{cases} 1, & \text{if } B_i \in \mathcal{F}(B_j), \\ 0, & \text{else.} \end{cases}$$

  k-periodic points of $\mathcal{F}$ $\leftrightarrow$ nonzero diagonal entries of P^k;

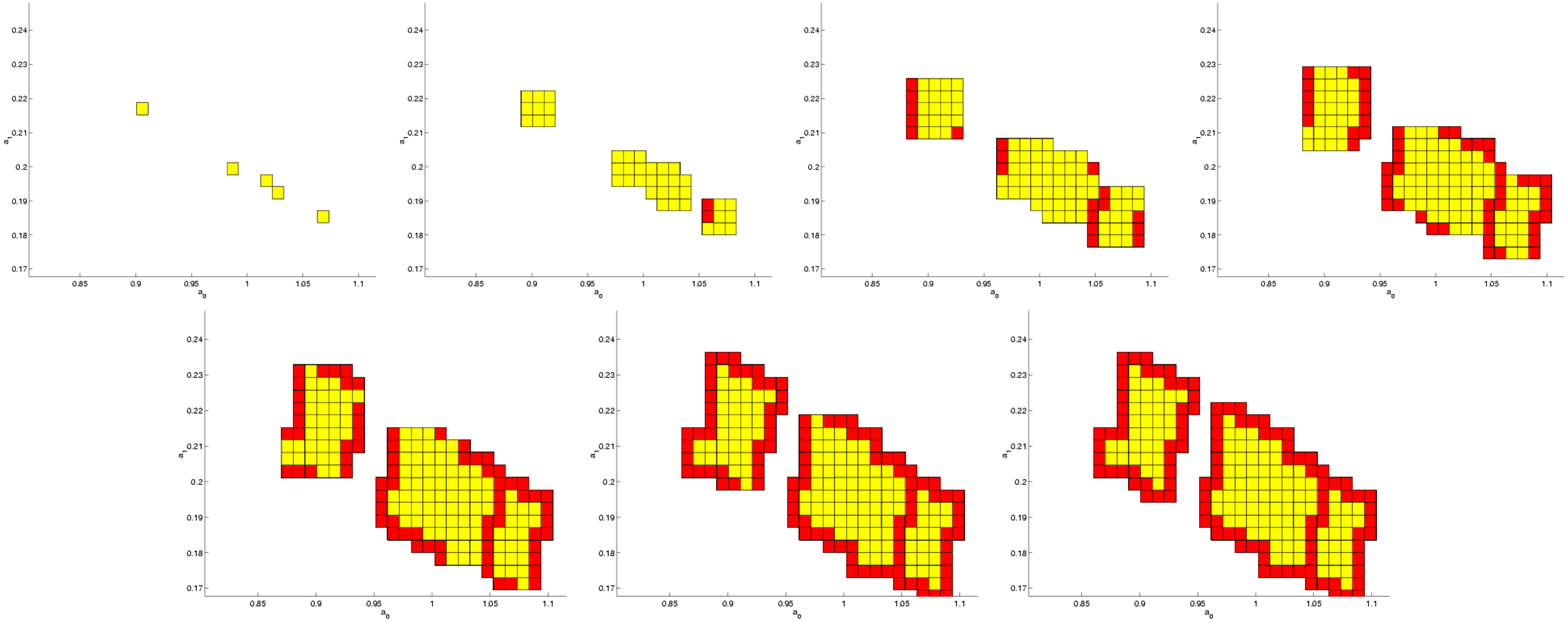
- Consider the graph $G = (\mathcal{B}, V)$,

$$V = \{(B, B') : B' \in \mathcal{F}(B)\}.$$

  - recurrent sets of $\mathcal{F}$ $\leftrightarrow$ strongly connected components of G;
  - connecting orbits of $\mathcal{F}$ $\leftrightarrow$ shortest paths (Dijkstra's algorithm);

- P and G are typically *sparse* and can be stored explicitely.

Turning the guess into a true isolating nbhd



Computing $\mathcal{F}$

- Heuristic approach: choose a finite set $T \subset B \in \mathcal{B}$ of *test points* in each box and set

$$\mathcal{F}(B) := \{B' \in \mathcal{B} \mid f(T) \cap B' \neq \emptyset\}.$$

- Note that since $\mathcal{B}$ is stored in a binary tree the complexity of this approximation of $\mathcal{F}(B)$ is only $O(\#T \cdot \log(\#\mathcal{B}))$.

- Rigorous approach:

  – Write

$$f(x+h) = f(x) + Df(x)h + f^{nl}(x,h).$$

  – For the box $B = B(c,r) \in \mathcal{B}$ (c: center, r: radius) compute $\varepsilon^{nl}(c)$ such that

$$\max_{|h| \leq r} \left| f^{nl}(c,h) \right| \leq \varepsilon^{nl}(c)$$

– For $x \in B$ set

$$F^{(m)}(x) = B(f(c), |Df(c)|r + \varepsilon^{nl}(c) + \varepsilon^{(m)})$$

– Finally define

$$\mathcal{F}(B(c,r)) = \{B' \in \mathcal{B} \mid F(c) \cap B' \neq \emptyset\}.$$

– Note: the set $\mathcal{F}(B)$ can be determined by a single depth first search of the tree:

$\mathcal{F}$ = **cap**(B, C, k)
    if $B \cap C \neq \emptyset$
        if depth$(B) = k$
            $\mathcal{F} := \mathcal{F} \cup \{B\}$
        else
            $\mathcal{F} := \mathcal{F} \cup \texttt{cap}(B^+, C, k) \cup \texttt{cap}(B^-, C, k)$
    return $\mathcal{F}$

- control of round off via interval arithmetic (BIAS, Profil, b4m, GAIO);

Lifting to the full system

- The compact set $Z = W \times V \subset L^2$ is of the form

$$Z = \prod_{k=0}^{\infty} [a_k^-, a_k^+].$$

- So far we computed an isolating neighborhood $I^{(m)} \subset W$ for $f^{(m)}$.

- **Theorem 2** . *If $I^{(m)}$ is an isolating neighborhood for $f^{(m)}$ and if*

$$f_k(Z) \subset (a_k^-, a_k^+), \quad k \geq m,$$

then

$$I = I^{(m)} \times \prod_{k=m}^{\infty} [a_k^-, a_k^+]$$

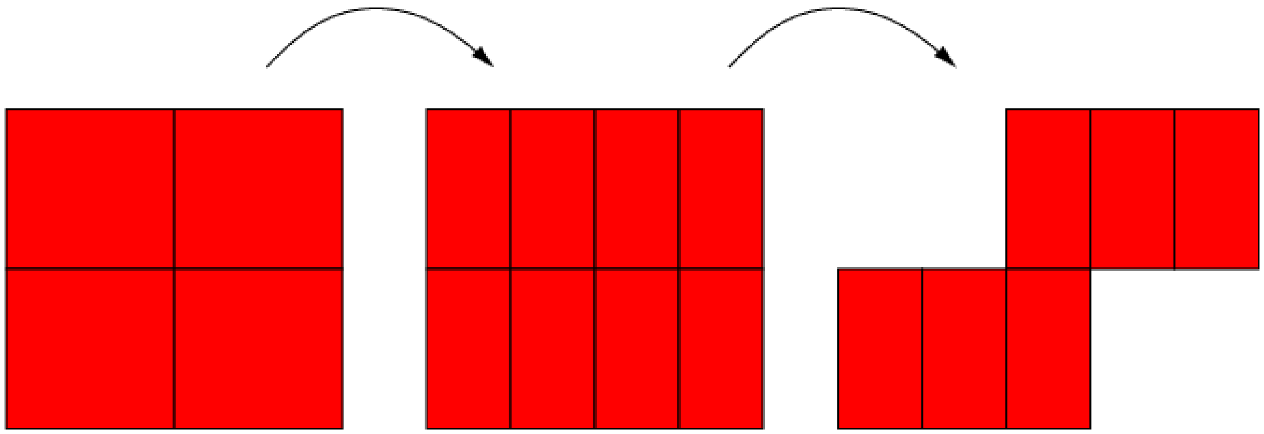
is an isolating neighborhood for Φ. In particular, the Conley index for a corresponding index pair is the same as for $I^{(m)}$.

Tightening isolating neighborhoods

Algorithm 1 (Dellnitz, Hohmann, 97) *Given the initial collection* $\mathcal{B}_0$*, one inductively obtains* $\mathcal{B}_k$ *from* $\mathcal{B}_{k-1}$ *for* $k = 1, 2, \ldots$ *in two steps.*

1. Subdivision: *Construct a new collection* $\widehat{\mathcal{B}}_k$ *by bisecting each box in* $\mathcal{B}_{k-1}$ *with respect to some coordinate direction.*
2. Selection: *Compute the relevant subset* $\mathcal{B}_k$ *of* $\widehat{\mathcal{B}}_k$*, i.e. set*

$$\mathcal{B}_k = \mathrm{Inv}(\widehat{\mathcal{B}}_k, \mathcal{F}).$$



Tightening the infinite tail

- For $k = m, \ldots$ compute an interval I_k, such that

$$f_k\left(\prod_{k=0}^{\infty}[a_k^-, a_k^+]\right) \subset I_k$$

  and set

$$[a_k^-, a_k^+]_{new} := I_k.$$

- Consider a *polynomial nonlinearity*

$$c(x)a(x)^p$$

  in Φ.

  The corresponding terms in the countable system read

$$a_k \mapsto \sum_{n_0,\ldots,n_{p-1}\in\mathbb{Z}} c_{n_0} a_{n_1} \ldots a_{n_{p-1}} a_{k-(n_0+\cdots+n_{p-1})}.$$

- *Regularity assumptions.* Suppose

$$|a_k| \le \frac{A}{s^{|k|}}, \quad |b_k| \le \frac{B}{b^{|k|}}, \quad |c_k| \le \frac{C}{s^{|k|}}, \quad k \in \mathbb{Z},$$

  for some constants $A, B, C > 0, b, s > 1$. Choose β such that $b/s < \beta < b$.

- One gets

$$|\sum_{n_1,\ldots,n_{p-1}\in\mathbb{Z}} c_{n_0} a_{n_1} \ldots a_{n_{p-1}} a_{k-(n_1+\ldots+n_{p-1})}| \le \frac{\alpha^p A^p C}{s^{|k|}} (\frac{b}{\beta})^{|k|}$$

  for some $\alpha = \alpha(s, b, \beta)$.

- For $k \ge M$ set

$$[a_k^-, a_k^+]_{new} := \frac{\alpha^p A^p BC}{(\beta s)^k}[-1, 1].$$

Increasing m

- *Problem:* For a fixed (small) m one gets stuck in tightening after a few steps, because the error $\varepsilon^{(m+)}$ is essentially fixed.

- *Solution:* Increase m. For the current collection $\mathcal{B}_k = \mathcal{B}_k^{(m)}$ set

$$\mathcal{B}_k^{(m+1)} = \left\{ B \times [a_m^-, a_m^+] : B \in \mathcal{B}_k^{(m)} \right\}.$$

and define $\mathcal{F}^{(m+1)} : \mathcal{B}_k^{(m+1)} \rightrightarrows \mathcal{B}_k^{(m+1)}$ suitably (via $F^{(m+1)}$).

- **Theorem 3** . *If $\mathcal{I}^{(m)}$ is an isolating neighborhood for $\mathcal{F}^{(m)}$ and if*

$$f_m(Z) \subset (a_m^-, a_m^+),$$

then

$$\mathcal{I}^{(m+1)} = \{B \times [a_m^-, a_m^+] : B \in \mathcal{I}^{(m)}\}$$

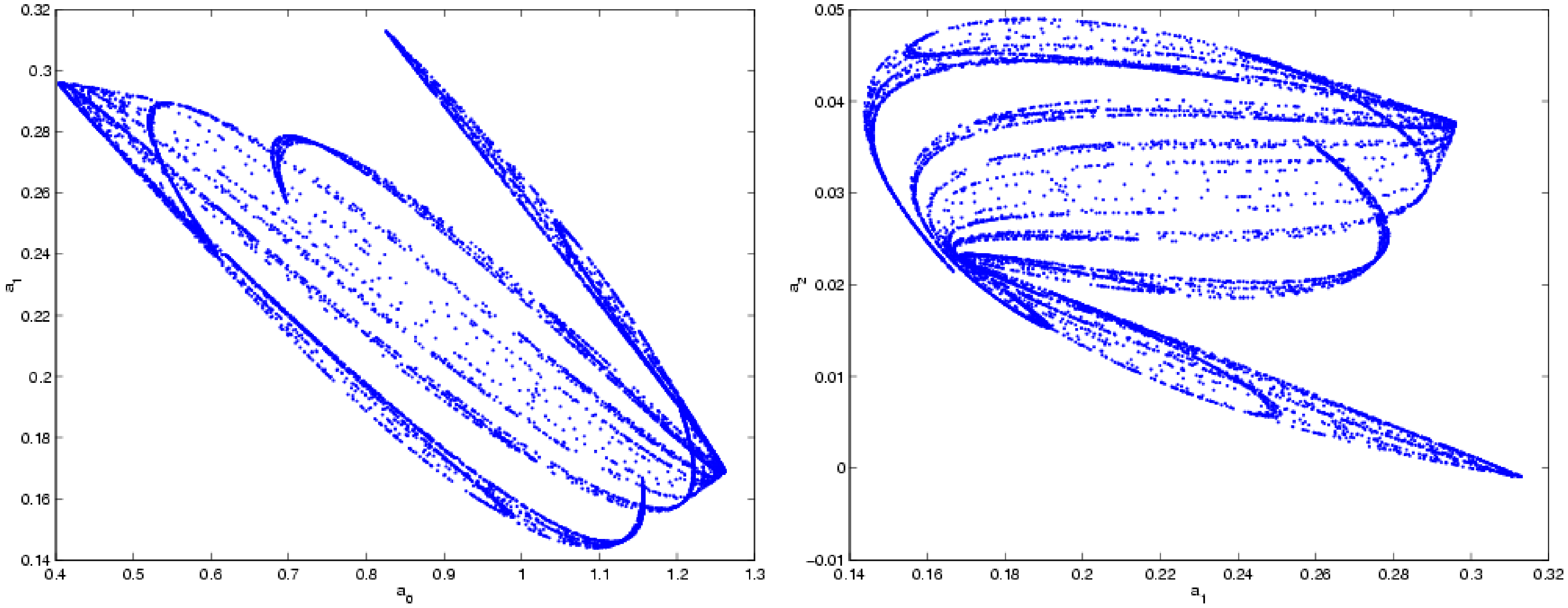
is an isolating neighborhood for $\mathcal{F}^{(m+1)}$.

Example computation

We consider the following parameters

$$\mu = 3.5, \quad b_k = 2^{-k}, \quad c_0 = 0.8, \quad c_1 = -0.2 \quad \text{and} \quad c_k = 0 \text{ for } k > 1.$$

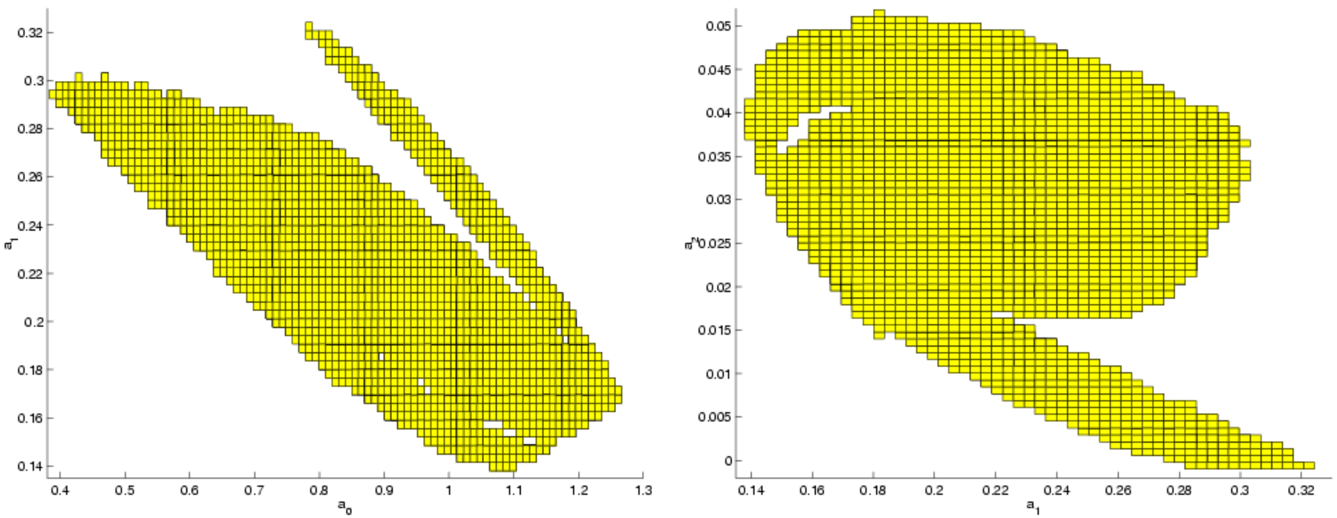
1. Running a simulation for $m = 50$:



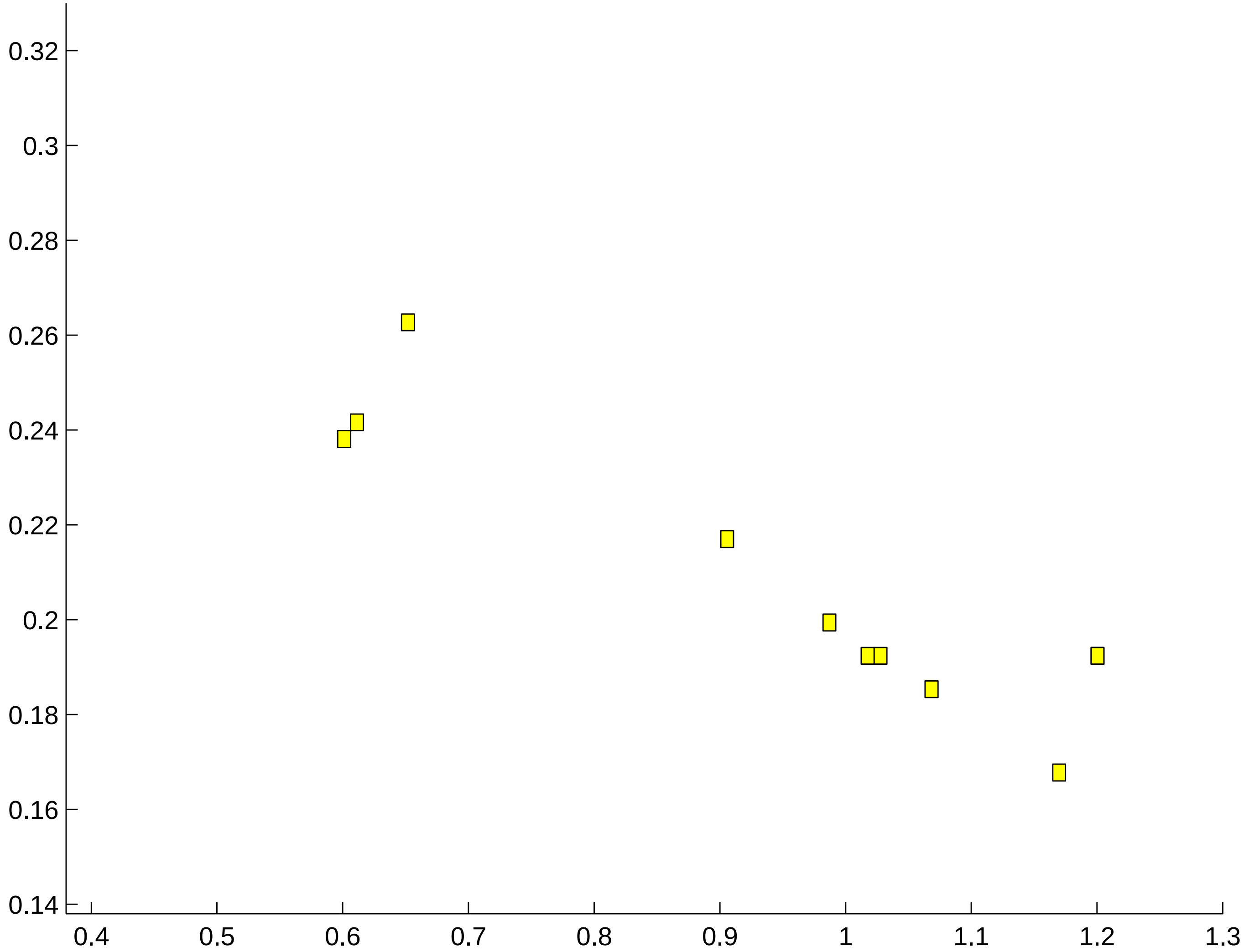
2. $\rightsquigarrow$ exponential estimate for the a_k with $A = 1$ and $s = 2$; initial bounds

k	a_k^-	a_k^+
0	0.2	1.5
1	0.05	0.5
2	-0.001	0.1
$2 < k < M$	-2^{-k}	2^{-k}

3. Computing a covering of the maximal invariant set in the chosen region:



4. Guessing invariant sets:



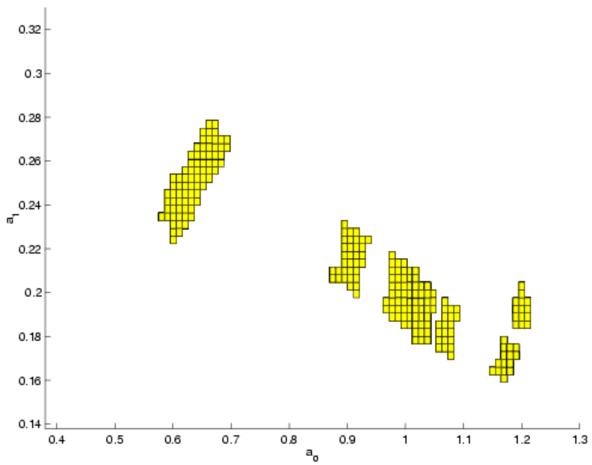
5. The nonlinear error for a box $B(c,r)$:

$$\varepsilon_k^{(m),nl}(c) = |\mu b_k| \sum_{j=-J}^{J} |c_j| \sum_{\ell \in L(m,k,j)} r_\ell r_{k-\ell-j}, \qquad k = 0,\dots,m-1.$$

6. Updating the bounds $a_k^{\pm}$:

$$\varepsilon^{(m+)} < (0.1, 0.2, 0.6, 2, 5)^T \cdot 10^{-5};$$

7. Combinatorial isolating neighborhood for $\mathcal{F}^{(m)}$:



8. Homology of the corresponding index pair:

$$H_*(N_1, N_0) \cong (0,\ \mathbb{Z}^8,\ 0,\ 0,\ \ldots)$$

and the map in homology:

$$F_1 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Theorem 4 *The map Φ possesses a heteroclinic orbit*

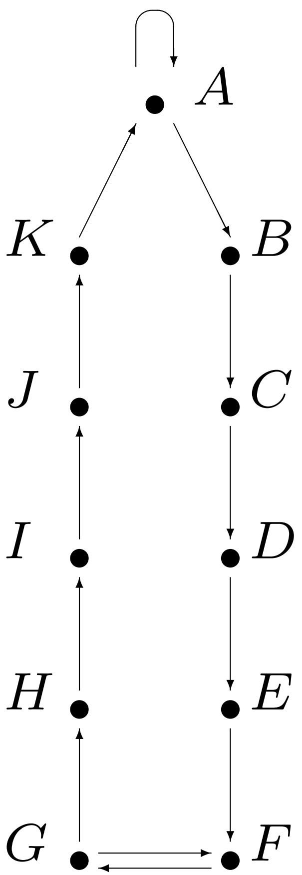
$$(a_j)_{j\in\mathbb{Z}}, \quad a_j \in L^2([-\pi,\pi]),$$

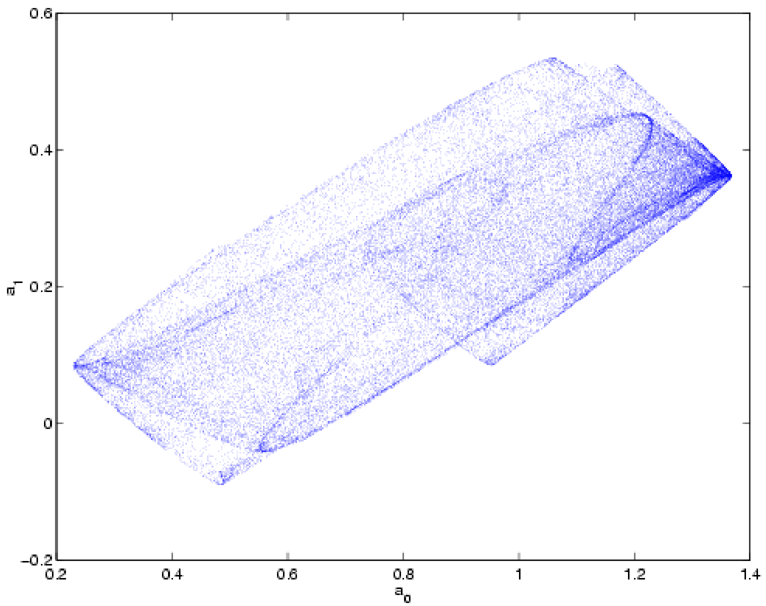
connecting a fixed point $p_1 \in L^2([-\pi,\pi])$ of Φ to a period two point $p_2 \in L^2([-\pi,\pi])$ of Φ, such that for the coordinates $(p_1),(p_2)$ and (a_j), $j \in \mathbb{Z}$,

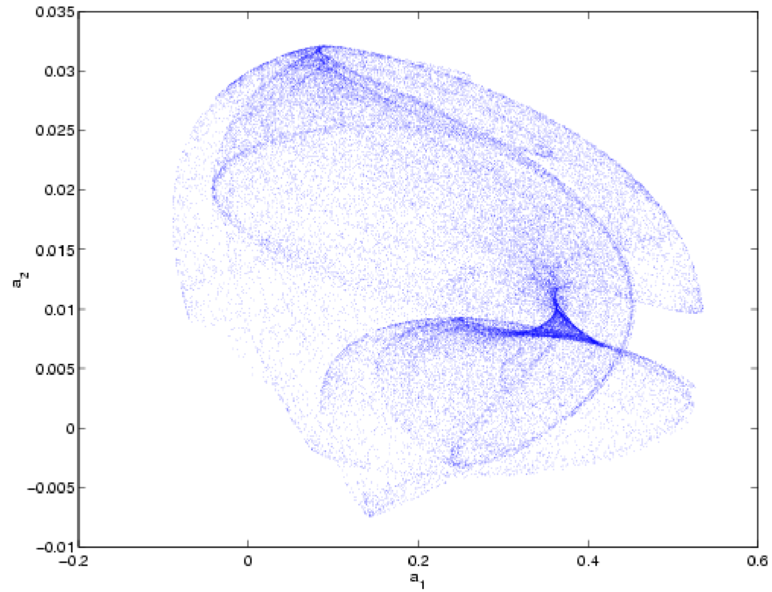
$$(p_1),(p_2),(a_j) \in |\mathcal{I}^{(12)}| \times \prod_{k=12}^{49} [a_k^-, a_k^+] \times \prod_{k=50}^{\infty} \frac{1}{2^k}[-1,1], \quad j \in \mathbb{Z}.$$

Here the $a_k^\pm$ are the final bounds.

2. Example computation







Theorem. For the parameter values [...] there is an invariant set, contained in [...], on which Φ is semi-conjugate to the subshift given by the transition graph.

Software

- CHomP — Computational Homology Program

  `http://http://www.math.gatech.edu/~chom/`

  Tomasz Kaczynski, Konstantin Mischaikow, Marian Mrozek, Pawel Pilarczyk.

- GAIO — Global analysis of invariant objects

  `http://www.upb.de/math/~agdellnitz/gaio`

  Michael Dellnitz, O.J.

- Scripts for these computations:

  `http://www.upb.de/math/~junge/kot_schaffer/code`