Tomasz Kapela

Computer assisted proofs of choreographies existence

Bedlewo, 4-10 June 2006

N-body Problem

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where $q_i \in \mathbf{R}^n$**.**

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where $q_i \in \mathbf{R}^n$**.**

We assume that:

- $m_i = 1$
- $n = 2 \implies q_i = (x_i, y_i),\ \dot{q}_i = p_i = (\dot{x}_i, \dot{y}_i)$

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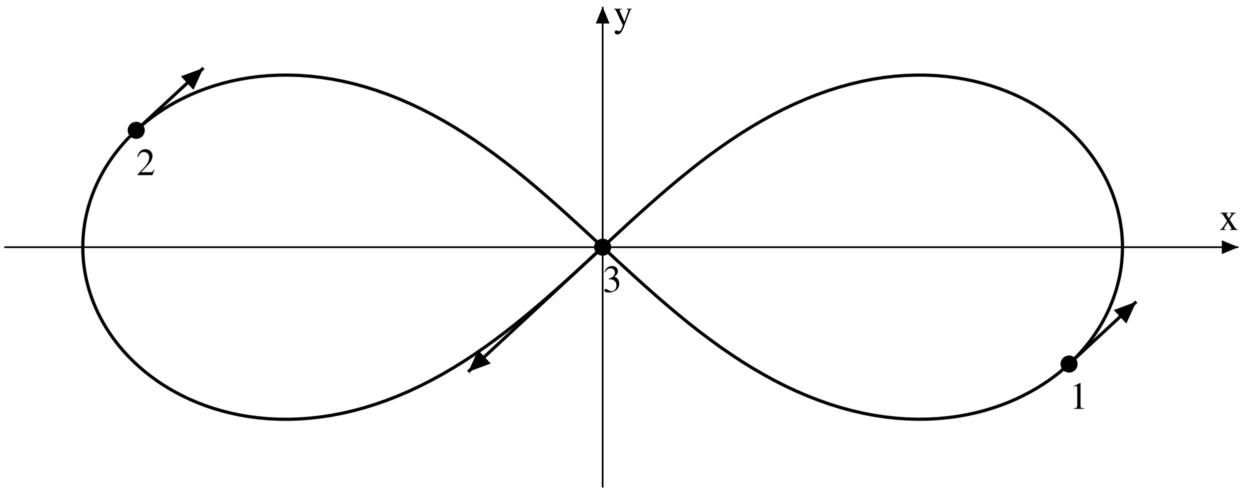
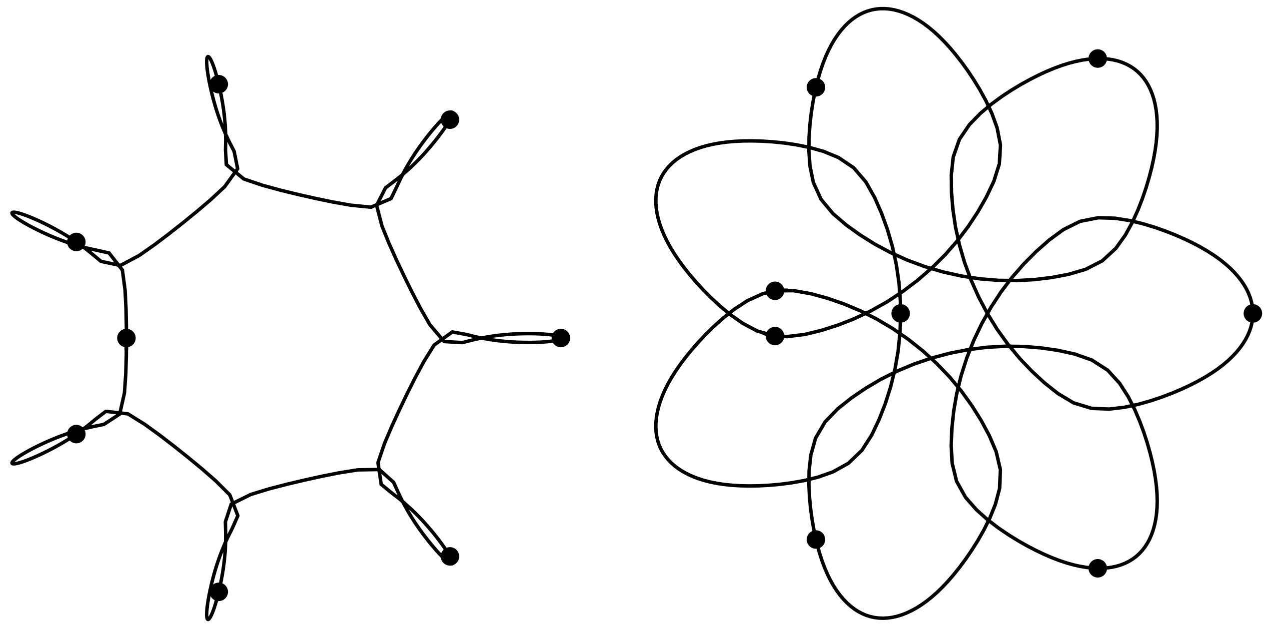
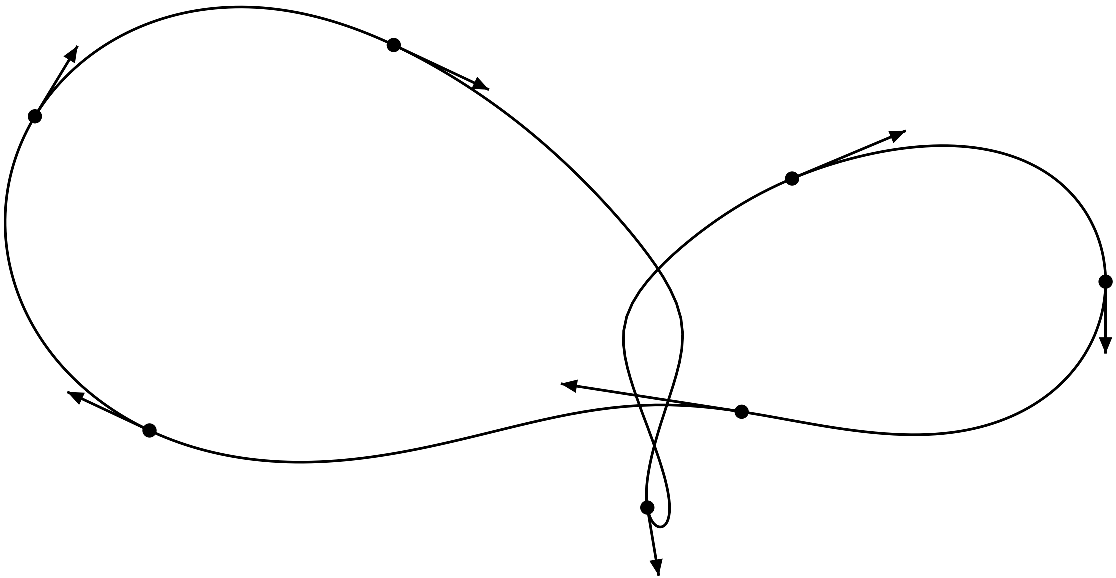


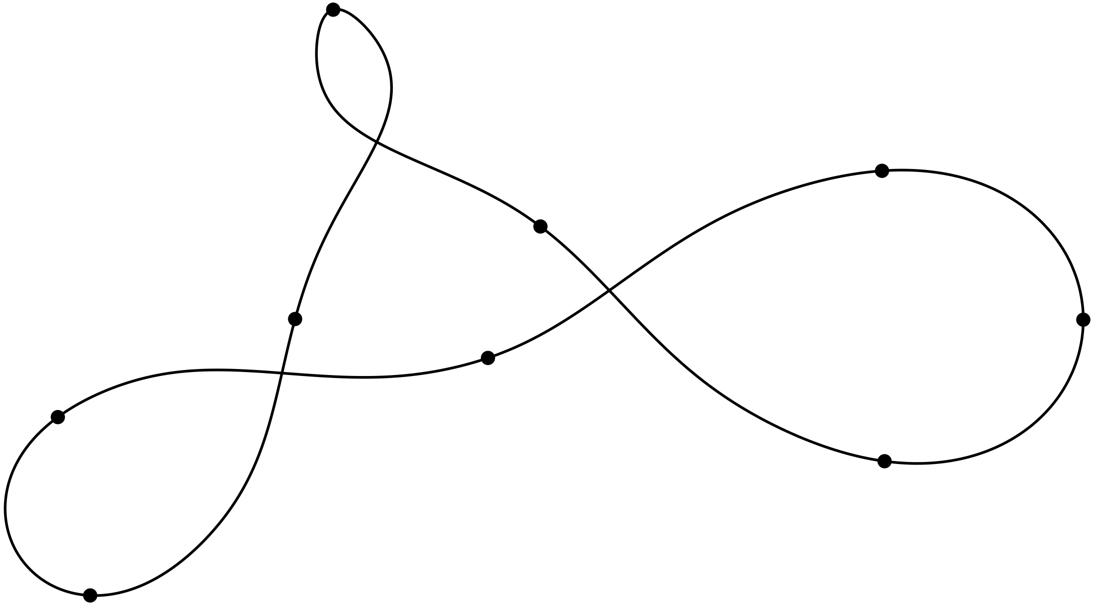
Famous eight shaped periodic orbit - The Eight



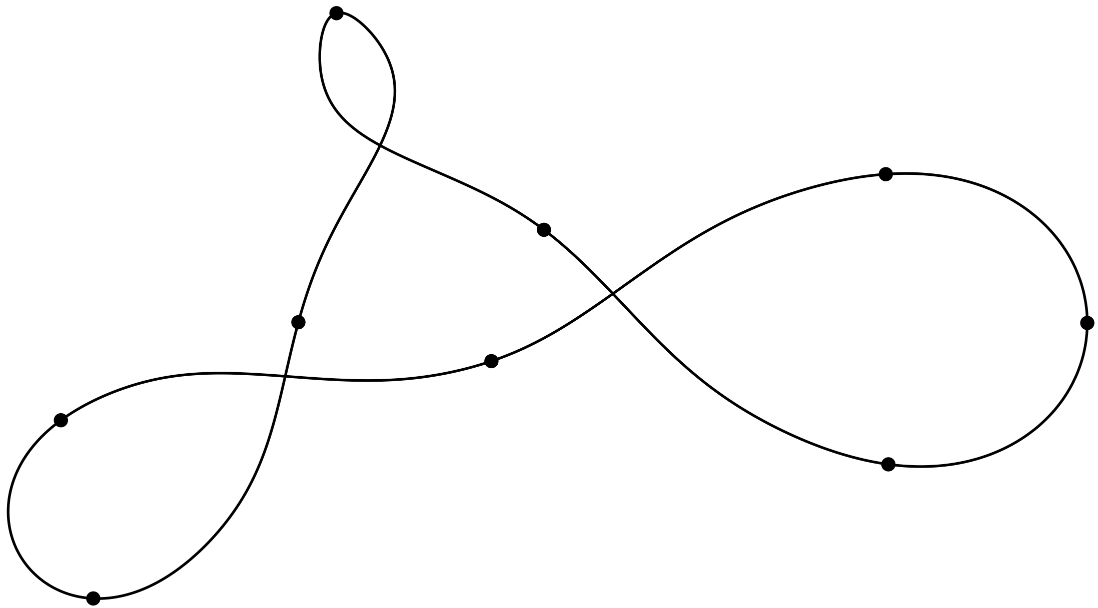
Choreographies of eight bodies



7 bodies - no symmetry



9 bodies - no symmetry



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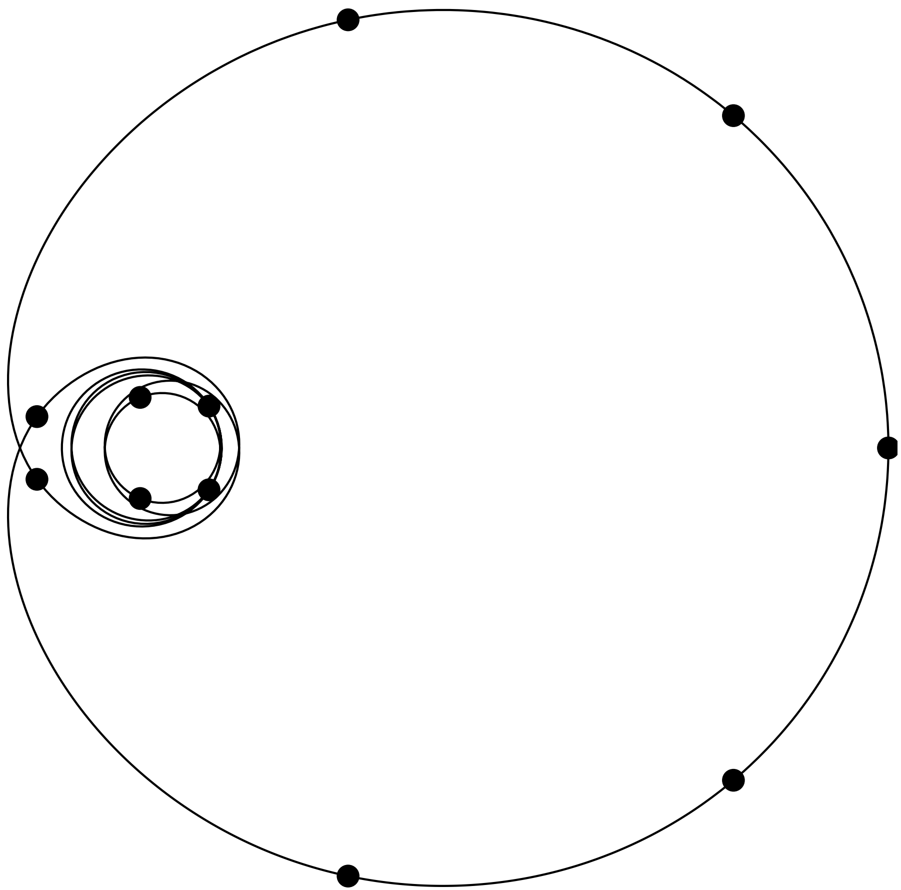

11 bodies

Proofs of existence of choreographies using variational methods:

- **Chenciner and Montgomery - The Eight,**
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How to use numerical data of the choreography to give a rigorous computer assisted proof of its existence?

We are searching for T periodic curve:

$q : \mathbf{R} \longrightarrow \mathbf{R}^2$

such that

- $q_k(t) = q(t + (k-1)\frac{T}{N})$ **is the position of k-th body,** $k = 1, 2, \ldots, N$

- $(q_1(t), q_2(t), \ldots, q_N(t))$ **is solution of the N-body problem.**

$x = (q_1, p_1, q_2, p_2, \ldots, q_N, p_N)$ - **point in the phase space**

$\varphi(x,t)$ - **flow defined by N-body equation**

$\sigma(x)$ **shift of particles** $q_1 \to q_2 \to \cdots \to q_N \to q_1$

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$\mathbf{F}(\mathbf{x_0}) = \mathbf{0} \Leftrightarrow \mathbf{x_0}$ initial condition for choreography.

Interval Krawczyk method

We assume that:
$F : \mathbf{R}^n \longrightarrow \mathbf{R}^n$ is a C^1 function,
$X \subset \mathbf{R}^n$ is an interval set,
$\bar{x} \in X$
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The **Krawczyk operator** is given by

$$\mathbf{K}(\bar{\mathbf{x}}, \mathbf{X}, \mathbf{F}) := \bar{\mathbf{x}} - \mathbf{C}\mathbf{F}(\bar{\mathbf{x}}) + (\mathbf{Id} - \mathbf{C}[\mathbf{DF}(\mathbf{X})])(\mathbf{X} - \bar{\mathbf{x}}).$$

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If $K(\bar{x}, X, F) \subset intX$, then $\exists! x^* \in X$ such that $F(x^*) = 0$.

If $K(\bar{x}, X, F) \cap X = \emptyset$, then $F(x) \neq 0$ for all $x \in X$

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Reduction due to the center of mass condition.

$$
\begin{cases}
\ddot{q}_i = \displaystyle\sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{(q_j - q_i)}{{q_i - q_j}^3} & \text{for } i = 1, 2, \ldots, N-1, \\
\text{where } q_N = -\displaystyle\sum_{i=1}^{N-1} q_i, & p_N = -\displaystyle\sum_{i=1}^{N-1} p_i.
\end{cases}
$$

We assume that at time $t = 0$ the first body is on the X axis with the velocity orthogonal to that axis

$$\mathbf{x} \quad = \quad (q_1, p_1, q_2, p_2, \ldots, q_{N-1}, p_{N-1})$$

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$$\begin{aligned} \mathbf{x} &= (q_1, p_1, q_2, p_2, \ldots, q_{N-1}, p_{N-1}) \\ &= (\mathbf{x_1}, \mathbf{y_1}, \dot{\mathbf{x}}_\mathbf{1}, \dot{\mathbf{y}}_1, \mathbf{q_2}, \mathbf{p_2}, \ldots, \mathbf{q_{N-1}}, \mathbf{p_{N-1}}) \end{aligned}$$

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&= (\mathbf{z}, \mathbf{c})
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So at time $t = 0$ we have $c = c_0 = (0, 0)$.

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Projections:

$$\pi_z(z, c) = z \qquad \pi_c(z, c) = c.$$

$\bar{\varphi}(x,t)$ **a flow generated by the reduced system**

$$\bar{\sigma}(x) = \left(-\sum_{i=1}^{N-1} q_i, -\sum_{i=1}^{N-1} p_i, q_1, p_1, \ldots, q_{N-2}, p_{N-2}\right)$$

$\bar{\varphi}(x,t)$ **a flow generated by the reduced system**

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$$\bar{\mathbf{G}}(\mathbf{x}) = \bar{\sigma} \circ \bar{\varphi}\left(\mathbf{x}, \frac{\mathbf{T}}{\mathbf{N}}\right)$$

$$\hat{\mathbf{F}}(\mathbf{z}) = \pi_{\mathbf{z}} \bar{\mathbf{G}}(\mathbf{z}, \mathbf{c_0}) - \mathbf{z}$$

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Function of the first integrals

$$\mathbf{J(x) \quad = \quad \left(\sum_{i=1}^{N} \frac{\|p_i\|^2}{2} - \sum_{1 \le i < j \le N} \frac{1}{\|q_i - q_j\|}, \quad \sum_{i=1}^{N} q_i \times p_i \right)}$$

Theorem 1. ***Let Z and C be two interval sets such that $z_0 \in Z$, $[\pi_c(\bar{G}(Z, c_0))] \subset C$ and $c_0 \in C$. If $K(z_0, Z, \hat{F}) \subset \operatorname{int} Z$ and an interval matrix $\left[\frac{\partial J}{\partial c}(Z, C)\right]$ is invertible, then there exists a zero of the map F.***

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Sketch of the proof:

$K(z_0, Z, \hat{F}) \subset \operatorname{int} Z \Longrightarrow \exists z^* : \hat{F}(z^*) = 0$

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$$J(z^*, c_0) = J(\bar{\sigma} \circ \bar{\varphi}\left((z^*, c_0), \tfrac{T}{N}\right)) = J(z^*, c_1)$$

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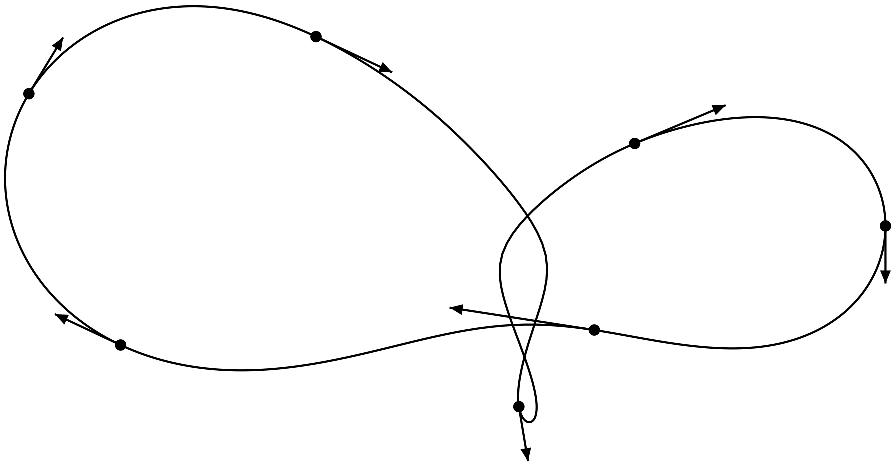
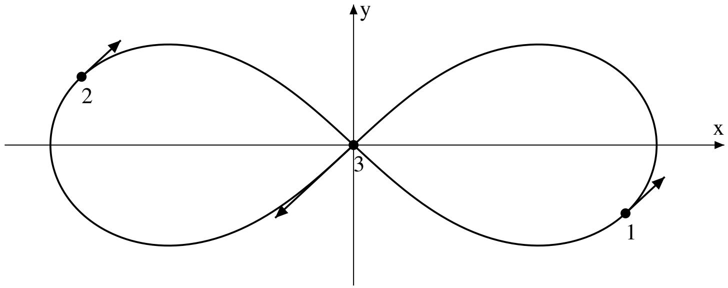
$c_0, c_1 \in C \Longrightarrow c_0 = c_1$

Theorem 2. *Nonsymmetric choreography with 7 bodies exists.*



Initial values	
Z	$\bar{z} + [\text{-2e-8},\ \text{2e-8}]^{70}$
Parameters of computations	
$\hat{F}(\bar{z})$ $\hat{F}(Z)$	time step : 0.0009, order of Taylor method : 13 time step : 0.000075, order of Taylor method : 7
Results of computations	
$\max\ \text{diam}\, K(\bar{z}, Z, \hat{F})$ $det\left(\frac{\partial \hat{J}}{\partial c}(Z, C)\right)$	7.08673242311641e-009 [1.67791,1.67791]
Computation time	$\hat{F}(\bar{z})$: 13.3 min., $\hat{F}(Z)$: 69 min.

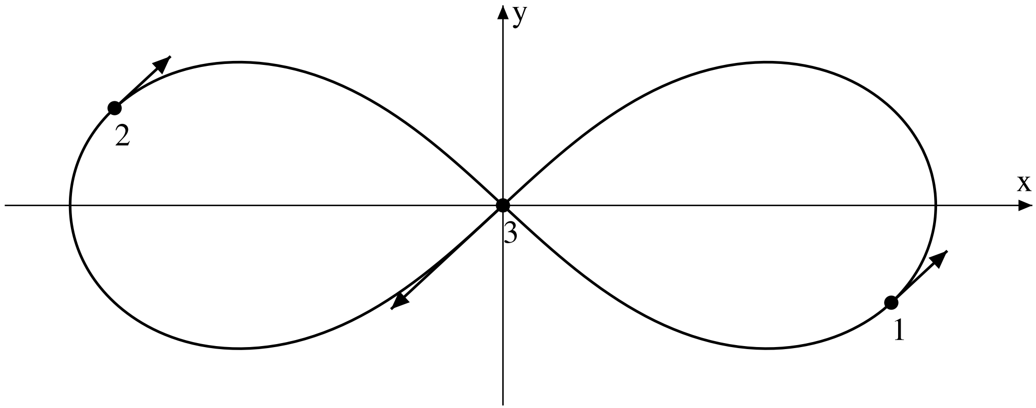


Initial values	
$\bar{z}$	$(1.07614373351,\ 0.46826621840,\ -0.53807186675,$ $-0.34370682775,\ -1.09960375207,\ -0.23413310920)$
Z	$\bar{z} + [\text{-1e-8},\ \text{1e-8}]^6$
$K(\bar{z}, Z, \hat{F})$	$(1.07614373_{447}^{223},\ 0.46826621_{895}^{793},\ -0.53807186_{370}^{965},$ $-0.34370682_{706}^{835},\ -1.09960375_{060}^{371},\ -0.2341331_{019}^{166})$
$\mathrm{diam}\, K(\bar{z}, Z, \hat{F})$	(2.23e-9, 1.02e-9, 5.93e-9, 1.28e-9, 3.10e-9, 1.46e-8)
$det(\frac{\partial J(Z,C)}{\partial c})$	0.4554_{99}^{72}

Using symmetry of the orbit to reduce computational cost.

Symmetry of the orbit is not only symmetry of a curve, but also involve time.

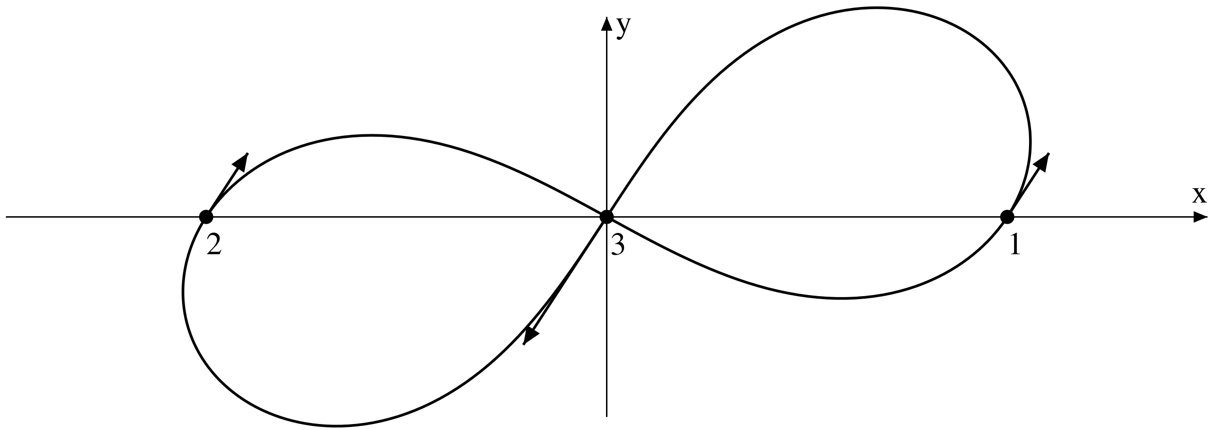
Symmetries of the Eight



$$q(t + \frac{T}{2}) = S_y q(y) \quad (1)$$

$$q(-t + \frac{T}{2}) = S_x q(t). \quad (2)$$

Expansion from the reduced space to the full space



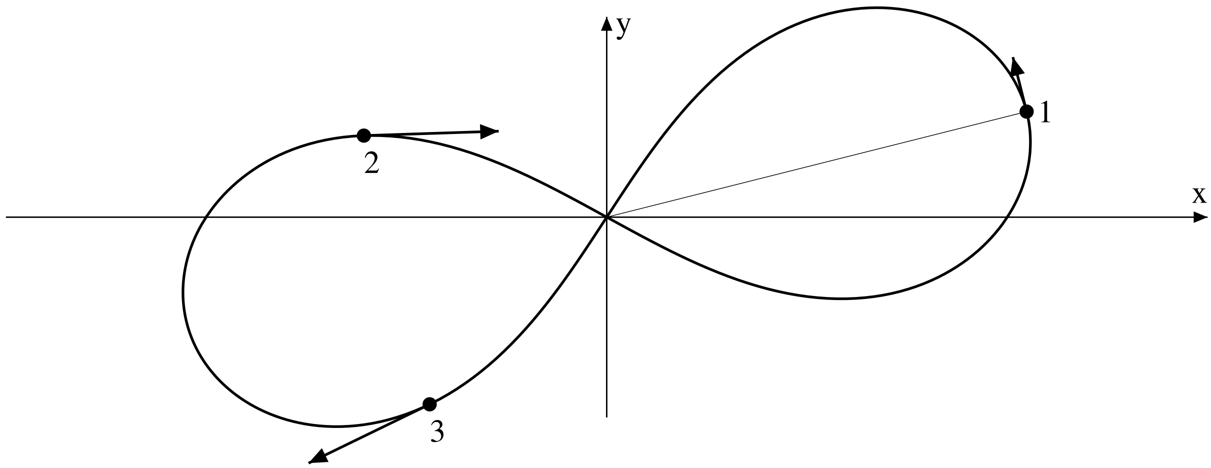
Rotated Eight - initial position

$$E(v,u) = (1,0,-1,0,0,0,v,u,v,u,-2v,-2u).$$

Poincaré map $P : \mathbf{R}^{12} \supset \Omega \longrightarrow \mathbf{R}^{12}$ **defined by section**

$$q_1 \cdot \dot{q}_1 = 0$$

Reduction from full space to the reduced space



Rotated Eight - final position

$$R(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3) = (\|q_2 - q_1\|^2 - \|q_3 - q_1\|^2, (\dot{q}_2 - \dot{q}_3) \times q_1),$$

$$\Phi = R \circ P \circ E.$$

There exists a locally unique $(v, u) \in \mathbf{R}^2$ **that** $\Phi(v, u) = (0, 0)$**.**

CONCLUSION:

- Algorithm gives general method for rigorous verification of choreography simulations

- 'brutal force' method not always gives results (e.g. increasing the order of the Taylor method can worsen the estimates)

- we need better algorithm for rigorous integration of ODE and methods of the interval sets representation

- multiple precision for intervals bounds can improve estimates and can make methods work for many orbits.