A computer assisted proof in one-dimensional dynamics

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Będlewo, 7 June 2006

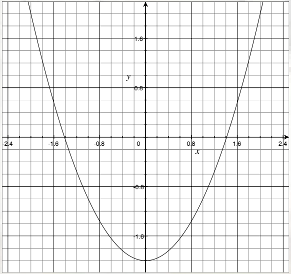
The quadratic/logistic family

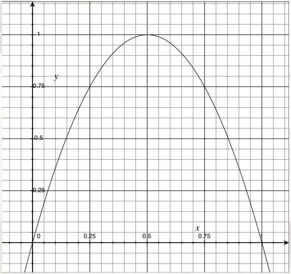
Quadratic family

$$f_a(x) = x^2 - a$$

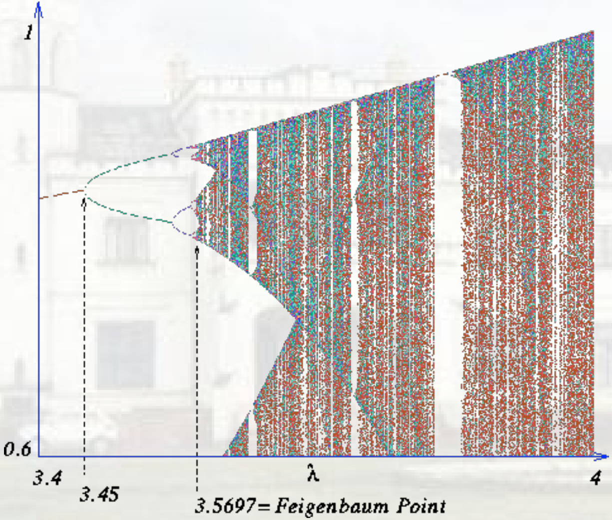
Logistic family

$$f_\lambda(x) = \lambda x(1-x)$$





Bifurcation diagrams



Regular dynamics

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Definition

f is regular (or periodic)

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f is regular (or periodic) if almost every point converges to a fixed or periodic orbit.

Stochastic dynamics

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f is **stochastic** if the dynamics of almost every point is described by a probability measure μ which is

- *invariant*
- *ergodic*
- *absolutely continuous wrt Lebesgue*
- has a *positive Lyapunov exponent:*

$$\int \log |f'| d\mu > 0.$$

Families of interval maps

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$$\{f_a\}_{a\in\Omega}$$

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The actual measures of Ω^- and Ω^+ are not known.

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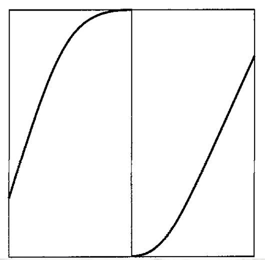

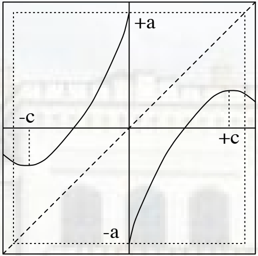

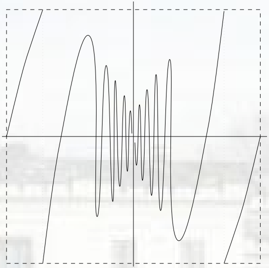

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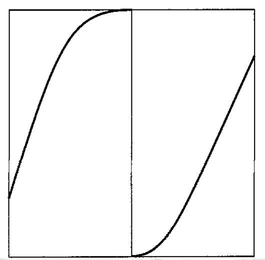

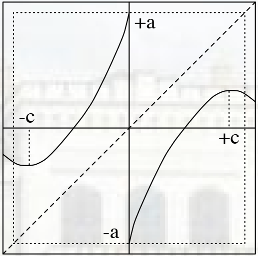

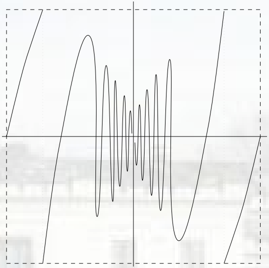

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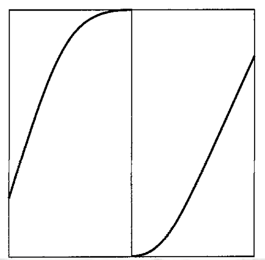

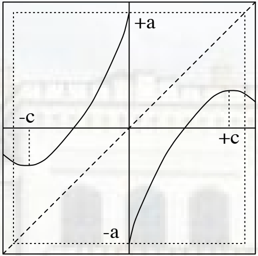

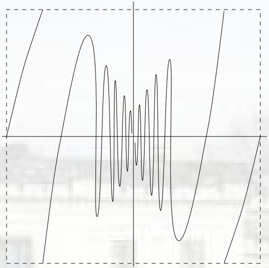

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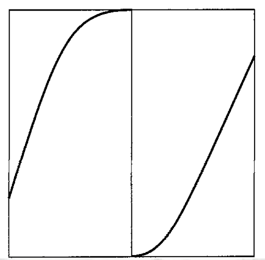

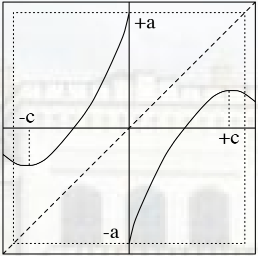

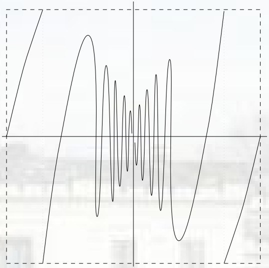

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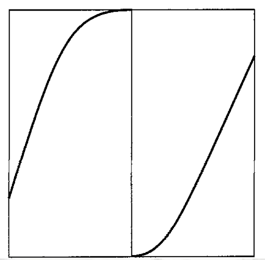

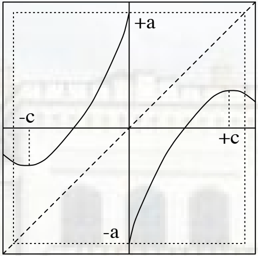

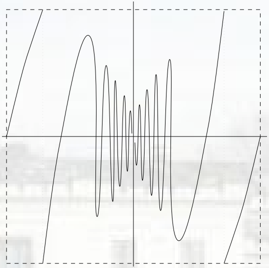

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Infinite-modal: Rovella-Pacifico-Viana ‘89, Araujo-Pacifico ‘05.

The basic question. . .

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$$\boxed{\text{Is } a \in \Omega^- \text{ ?} \qquad \text{Is } a \in \Omega^+ \text{ ?}}$$

Ω^-

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Simó and Tatjer [1991, 2006].

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Numerical evaluation of periodic windows.

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If $\Omega = [a_{feig}, 2]$

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If $\Omega = [a_{feig}, 2]$

$$\frac{|\Omega^-|}{|\Omega|} > 0.103\ldots$$

What happens in the remaining 90% of parameters ?

$a \in \Omega^+$

Theorem (Conditions which give $a \in \Omega^+$)

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Require computing an infinite number of iterations with infinite precision. *Impossible in practice*. Arbieto-Matheus [04] Ω^+ is undecidable.

The basic question revisited

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Let $a \in \Omega = [a_{feig}, 2]$.

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Possibly combine many neighbourhoods (in the complement of Simó-Tatjer set of periodic windows) to get global bound for

$$|\Omega^+|/|\Omega|$$

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Then

$$|\Omega^+| \geq 0.98|\Omega|$$

In particular

$$|\Omega^+| \geq 10^{-5000}.$$

Further work: theoretical part

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Extensions to more general families

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- multiple critical points

Further work: theoretical part

Extensions to more general families

- multiple critical points
- degenerate critical points

Further work: theoretical part

Extensions to more general families

- multiple critical points
- degenerate critical points
- critical points depending on the parameter

Further work: theoretical part

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Two kinds of issues:

- Numerical algorithms for the verification of conditions (A1)-(A4).
- Determining admissible sets of constants satisfying conditions (C1)-(C4).

Conditions (A1)-(A4)

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- with powerful computational methods to check assumptions in specific situations
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Then

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